A STUDY OF THE SPECTRAL CHARACTERISTICS OF PULSE-POSITION MODULATED WAVEFORMS

ARTHUR AUGUSTUS BERGMAN

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by

Arthur Augustus Bergman

Captain, United States Marine Corps

Submitted in partial fulfillment of
the requirements for the degree of

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IN
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ABSTRACT

The power spectral density of a pulse-position modulated (PPM) waveform is determined in part by certain statistics of the modulating process. In this paper a mathematical model is developed to determine the effects of three different modulating processes upon the power spectral density. The processes considered are the Gaussian, the exponential, and the uniform processes, with statistical independence assumed between the sample values that modulate the PPM waveform. The Gaussian process is also developed for the case in which statistical independence may not be assumed; that is, a correlation exists between the sample values of the modulating process. The model will also determine the power spectral density of pulse-amplitude modulated (PAM) waveforms, or for a combination of PAM and PPM.

The mathematical model is reduced to a computer model and computed results are displayed. The design of a pulse-position modulator is shown, and experimental measurements of power spectra are compared with the theoretically determined spectra. The theoretical determination of power spectral density and the actual measurement of spectra of finite-time samples are discussed.

The major portion of the research reported on was conducted by the writer during his Industrial Tour at Sylvania's Electronic Defense Laboratories (EDL) in Mountain View, California.

The writer gratefully acknowledges the helpful assistance and encouragement received from both his thesis advisor, Professor George H. Marmont, and from Dr. Paul O. Scheibe, formerly of EDL.

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TABLE OF SYMBOLS AND ABBREVIATIONS

Symbol	Definition
a	Simplifying constant; see page 27
A	Absolute value of the range of the modulating process x
b	Simplifying constant; see page 27
e	2.71828.....
E	The expected value, taken over an ensemble
f	Frequency, in cycles per second
$G_\epsilon(\omega)$	The characteristic function of ϵ
$G(j\omega_1, j\omega_2)$	The joint characteristic function of a bivariate process ϵ_1, ϵ_2
i	An integer index
I	The number of sample intervals beyond which the autocorrelation function is assumed to be zero [i.e., $R(kT_o) = 0$, $k > I$]
j	Defined such that $j^2 = -1$
k	An integer index
$K(\tau)$	The covariance function; see page 4
L	Proportionality constant indicating the range of deviation of a pulse permitted in a given interval T_o. For example, if the possible range of a pulse was $0.1T_o$ ($= T_o/L$), then $L = 10$
m	An integer index
n	An integer index
N	The limiting value of the indexing integers m and/or n
$p_{\epsilon_k}(\epsilon)$	The probability density function of the ϵ process, with the random variable denoted as ϵ_k

TABLE OF SYMBOLS AND ABBREVIATIONS

(Continued)

Symbol	Definition
pdf	Probability density function
PPM	Pulse-position modulation
$R(\tau);R^n(\tau)$	The autocorrelation function (time averaged) of a member of an ensemble of message processes
$S(f)$	The power spectral density (one sided), $f \geq 0$
$S'(f)$	The two sided power spectral density, $-\infty < f < \infty$
$S_U(f)$	The power spectral density of a typical sampling pulse
$S^n(f, T)$	The periodogram of the $n^{\underline{th}}$ member of an ensemble
T	The interval over which the signal (or message process) is defined
T_o	The sampling interval
u	The PPM pulse duration
$u_s(f)$	The unit-step function
$U(t)$	The pulse function (defines the shape of the "typical" pulse and exists for a time u).
$Var(\epsilon)$	The variance of the epsilon process
$x(t)$	The voltage waveform describing the message which is to modulate the PPM waveform
x_k	The sampled value of $x(t)$ at $t = kT_o$
$X_U(f)$	The Fourier transform of the pulse function $U(t)$
$y(t)$	The process for which the power spectral density is to be determined
$y^i(t)$	The $i^{\underline{th}}$ member of the ensemble $y(t)$

TABLE OF SYMBOLS AND ABBREVIATIONS

(Continued)

Symbol	Definition
y_n^i	The peak amplitude of the $i^{\underline{th}}$ member of the ensemble of the ensemble of PPM waveforms in the time interval $(nT_o + \epsilon_n) < t < (nT_o + \epsilon_n + u)$
$y_N^i(t)$	The $i^{\underline{th}}$ member of the ensemble of truncated pulse wave-forms $y_N(t)$
$Y^i(f)$	The Fourier transform of the time function representing the $i^{\underline{th}}$ member of the ensemble $y(t)$
α	Simplifying expression; see pages 18 and 32
$\delta(f)$	The Dirac delta function
ϵ	The process describing the pulse displacement (in time) from its unmodulated position in the PPM waveform
ϵ_k	The pulse displacement (in time) of the pulse corresponding to the sample instant $t = kT_o$
λ	Simplifying expression; see page 27
μ_y	The mean value of the process y, or $E\{y\}$
π	3.14159.....
ρ	The correlation coefficient
$\rho(\tau)$	The normalized autocorrelation function; $\rho = R(\tau)/R(0)$
$\rho(kT_o);\rho(k)$	The value of $\rho(\tau)$ at $\tau = kT_o$
σ_x	The standard deviation of the process x (RMS deviation about the mean)
σ_x^2	The variance of the process x; $\sigma_x^2 = E\{(x - \overline{x})^2\}$
σ_n	The standard deviation of the ϵ process normalized to a unit time interval; $\sigma_n = \sigma_\epsilon / T_o$
τ	Time variable
ψ	Simplifying expression; see page 27
ω	Angular frequency in radians per second; $\omega = 2\pi f$

1. Introduction

The objective of this report is to relate a theoretical approach for the determination of power spectral density, to show computations of power spectral densities performed for a particular class of signals, and to compare experimental measurements of the power spectral densities to that predicted by theory.

The power spectral density of a pulse-position modulated (PPM) waveform typically consists of a continuous component and discrete frequency components. The relative amount of the total power contained in each of the component types is determined by the modulation process. The distribution of power in the spectrum is determined by the modulation process, the pulse shape, and the average pulse repetition rate.

In this study, a mathematical model is developed which determines the power spectral density of PPM waveforms. The mathematical model is then reduced to a digital computer program. The computer program permits the computation of the power spectral density of a PPM waveform that is modulated by a uniform process, a Gaussian process, or an exponential process. The modulation process that results in a statistical correlation between pulses within the pulse train with respect to their displacements (in time) is developed for the Gaussian process only. Statistical independence between pulses is assumed for the other processes considered.

The pulse shape is assumed to be rectangular for the computer model. The pulse width and the average pulse repetition rate are variable

parameters of the program. The pulse amplitude is also a variable parameter, thereby allowing computations to be performed for combined pulse-position and pulse-amplitude modulated waveforms. By setting the pulse position modulation equal to zero, the computation of the power spectral density of a pulse waveform that is only pulse-amplitude modulated may also be accomplished with the computer model.

A discussion of the use of finite-time interval measurements, using a periodogram technique to give estimates of the power spectral density of a process, is included in this report. The periodogram technique is used to determine the power spectral density of a PPM waveform. Since the characteristics of the experimental PPM waveform are also used as an input to the computer model, a comparison of the theoretical and experimental results is included.

2. A Discussion of Spectral Analysis (General)

The purpose of a communication system is to convey information in some form from one place to another. The system would normally include a transmitter, a transmission medium, and a receiver which recreates a reasonable representation of the original message. The message creating the transmitted signal will most likely be one of a class of stochastic phenomena; that is, the message is a random function of time. Also, the noise accompanying the signal (introduced by the communication system) will normally be a stationary Gaussian random process. The analysis of such signals is difficult unless one resorts to the use of the statistical properties of the signals. The power spectral density furnishes us such an analysis technique, as it is related to the statistical parameters of the signal. In particular, the power spectral density is related to the covariance (or autocorrelation) function.

In the development of spectral analysis techniques, one normally considers the random signal to be a stationary process. A stationary process is one in which the statistical parameters of the signal are the same everywhere, independent of which long interval over which they are measured. A similar concept is to consider a process as an ensemble of signals (an ensemble is a set of similar things). If the statistical averages of the ensemble measured at one instant of time are the same for any other instant of time, then the process is also said to be stationary in time.

If a single process of the ensemble of signals has statistics over

a long period of time that are the same as the statistics of the ensemble at a given time, then the process is said to be "ergodic". An ergodic process will always be stationary, but a stationary process does not necessarily imply ergodicity [6].

In this paper all processes will be assumed stationary unless otherwise stated.

To arrive at the statistical parameter used to obtain the power spectral density, consider the random ergodic process $y(t)$. A typical member of the ensemble is $y^n(t)$. The ensemble averaged correlation function (covariance function $K(\tau)$) is defined as the second joint moment, or

$$K(\tau) = E\left\{y_t y_{(t+\tau)}\right\},$$

where E denotes the expected value taken over the ensemble.[1] The auto-correlation function $R^n(\tau)$ of a typical member of the ensemble is

$$R^n(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} y^n(t) y^n(t+\tau)\, dt$$

In general,

$$K(\tau) = E\left\{R^n(\tau)\right\},$$

where the $E\left\{R^n(\tau)\right\}$ here is the autocorrelation function averaged over the

[1]The statistical definition of covariance would require that the square of the mean $(\mu_y{}^2)$ be subtracted from the above expression. It was decided, however, to include the mean in this investigation, thus permitting a simplified development that will handle signals that possess a non-zero mean.

entire (infinite) ensemble. This average should normally be performed <u>before</u> the time average of $R(\tau)$.

However, for the ergodic case

$$R^n(\tau) = K(\tau) \quad \text{(probability 1)} .$$

The mathematics relating the power spectral density to the autocorrelation (or covariance) function were first developed by Wiener and Khintchine. The Wiener-Khintchine Theorem states that the power spectral density $S(f)$ is the Fourier transform of the covariance function, or

$$S(f) = 2\int_{-\infty}^{\infty} K(\tau)e^{-j\omega\tau}d\tau, \quad f \geq o$$

Note that in the above expression $S(f)$ is defined for positive frequencies only, hence the factor of two in the equation. This is obviously the more practical case, and is permissable since $S(f)$ is an even function (as is $R(\tau)$). One can obtain the autocorrelation function by taking the inverse Fourier transform of the power spectral density (again assuming ergodicity). Thus

$$R(\tau) = \frac{1}{2}\int_{o}^{\infty} S(f)\cos \omega\tau df \qquad \text{(probability 1)}$$

and

$$S(f) = 4\int_{o}^{\infty} R(\tau)\cos \omega\tau d\tau \qquad \text{(probability 1)}$$

It should be mentioned that while a signal has a unique power spectral

density, the converse is not true. All phase information in the signal is lost in S(f), thus the original signal could not in general be recovered from its power spectral density. Furthermore, a particular S(f) theoretically represents the power spectral density for a multitude of signals.

3. Finite-Time Interval Spectral Analysis

The developments of the previous section followed the typical theoretical approach to signal analysis in that infinite time intervals were assumed. In practical analysis situations, such intervals are of course always finite. For some processes, a finite investigation interval can be considered adequate if the interval is large with respect to the fluctuation time of the process under investigation. Since finite time measurements of spectra of PPM waveforms will be compared with the results from the mathematical model, a discussion of possible sources of error due to analysis of the finite-time interval observation of the signal is included.

The problem of finite time interval analysis of a stationary random process can be stated quite simply as follows. A sample function of the random process $y(t)$ is observed during an interval $t = -T/2$ to $t = T/2$. Will this observation contain enough of the characteristics of the process to give a reasonable estimate of the power spectral density of the process? Note that the random process under consideration is stationary. The stationarity of the signal becomes more meaningful (and necessary) when considering finite-time analysis.

To develop the theory for finite-time analysis, again consider the ensemble $y(t)$, except now let $y(t)$ vanish outside the interval $(-T/2, T/2)$. An amplitude spectrum for a member of the ensemble is

$$Y^n(f)_T = \int_{-\infty}^{\infty} y^n(t)_T e^{-j\omega t} dt = \int_{-T/2}^{T/2} y^n(t) e^{-j\omega t} dt$$

Now the finite-time power spectral density of the ensemble is

$$S(f)_T = E\left\{\frac{2}{T} Y^n(f)_T Y^n(f)_T^{\,*}\right\} = \frac{2}{T} E\left\{|Y^n(f)_T|^2\right\},$$

where * denotes the complex conjugate. The true power spectral density S(f) is defined in terms of the ensemble average over an infinite time interval (and is equivalent to the definition on page 5). Specifically,

$$S(f) = \lim_{T\to\infty} S(f)_T = \lim_{T\to\infty} \frac{2}{T} E\left\{|Y^n(f)_T|^2\right\}$$

where the ensemble average is carried out before the limit is taken.[1]

Again, consider the sample function of a single random process. The finite time transform is equivalent to $Y^n(f)_T$ of the preceding paragraph, and will now be used as an approximation to the Fourier transform of a single stationary process. A finite power spectral density of the single random signal may now be defined.

$$S^n(f, T) = \frac{2}{T} |Y^n(f)_T|^2$$

S(f, T) is called a "periodogram",[2,3] and has received considerable attention due to the fact that it is one type of practical measurement that can be made on real processes (signals). Observe that the power

[1] Middleton, D. An Introduction to Statistical Communication Theory. McGraw-Hill, 1960, 143

[2] Abramson, N. and Farison, J. On Statistical Communication Theory. Technical Report No. 2005-1, SEL-62-078, Stanford University, August 1962, 30-33

[3] Grenander, U. and Rosenblatt, M. Statistical Analysis of Stationary Time Series. John Wiley & Sons, 1957, 91-94

spectral density for the time limited ensemble is the expected value of the periodogram over the entire ensemble, or

$$S(f)_T = E\left\{ S^n(f, T) \right\}$$

and

$$S(f) = \lim_{T \to \infty} E\left\{ S^n(f, T) \right\}$$

Thus, one measure of the reliability of the periodogram is the fact that as $T \to \infty$, the mean value of the periodogram does equal the power spectral density. However, as shown by Abramson and Farison, for some very common random processes, the variance of $S(f, T)$ does not go to zero as $T \to \infty$.[2] The fact that the periodogram (being a random variable) has the desired mean value but may vary extensively about the mean for certain processes has caused some skepticism about its use as a spectral density estimator [5].

It has been shown that the periodogram is not a consistent estimate of the power spectral density [10]. However, the periodogram may be used to give consistent estimates of the power spectral density at specific frequencies f_i. This is accomplished by the use of weighting functions, or "spectral windows" about the frequency f_i. A possible method then of obtaining a consistent estimator of the power spectral density would be to have many adjacent windows, and at the end of the analysis time T observe the response of all windows simultaneously. A smoothing between the point estimates observed would naturally be desirable.

Periodogram calculators fall into three general classes; that is, the filtering, computing and smoothing may be performed in three distinctly different ways. These are

1. Parallel filter banks,
2. Sweeping analyzers,
3. The coherent memory delay integrator.

The parallel filter bank technique is, as the name implies, a bank of narrow-band filters, each with a common input (the signal to be analyzed). At the output of each filter is an energy detector, which is periodically sensed by a sweep circuit that is connected to the vertical deflection plates of an oscilloscope. The parallel filter bank technique is depicted in block diagram form in Figure 3-1.

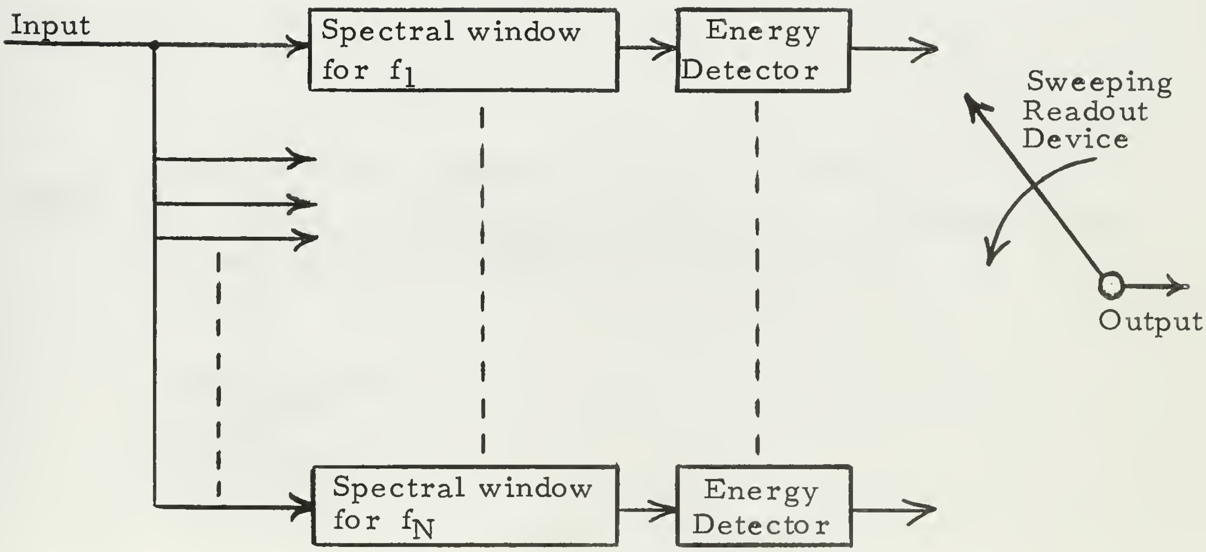


Figure 3-1. Parallel Filter Bank Periodogram Calculator.

The number of filters (N) may vary with the frequency range (analyzer bandwidth) and the resolution required. The bandwidth of a filter is

approximately the inverse of the rise time. Obviously, the bandwidth of the filters determines the resolution of the device. The "analysis time", or Period T, using this method is approximately the rise time of the filter. From this, one may observe that the resolution and the analysis time are essentially the inverse of each other. Consequently, if one designs an analyzer for fine resolution, then the signal must be observed for a longer period, and the converse is also true. This important relationship holds for periodogram calculators in general.

The sweeping analyzer is the more common general laboratory type of device. With this device, the input signal is mixed with a sweeping local oscillator, the output of which is filtered through a single "spectral window". The signal is then detected and displayed on an oscilloscope. A block diagram for the sweeping analyzer is shown in Figure 3-2.

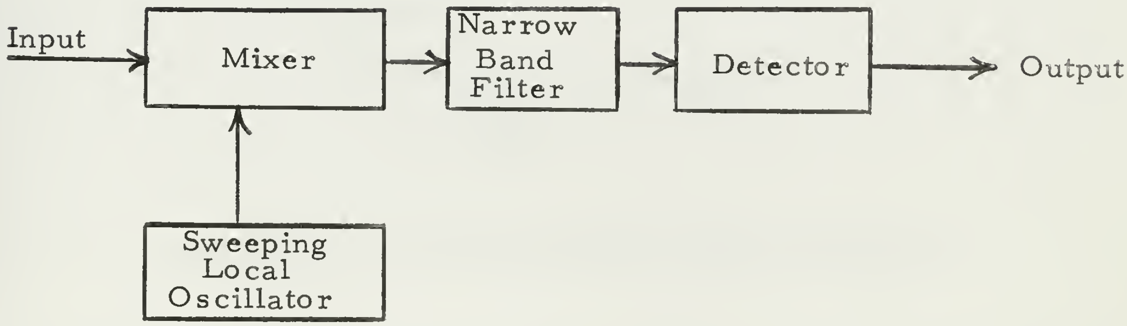


Figure 3-2. Sweeping Analyzer Periodogram Calculator.

A disadvantage of this type of analyzer is the adverse effect of the transient condition at the input to the filter while the local oscillator is

sweeping. If the sweep rate is too rapid, the filter may continue to ring due to this transient effect, even though the signal is no longer present at the input to the filter, and the resolution is no longer the bandwidth of the filter.

The third type of periodogram calculator to be discussed is the coherent memory delay integrator. This device is somewhat sensitive in its operation, but the concept is unique and merits consideration. The block diagram of the device is shown in Figure 3-3.

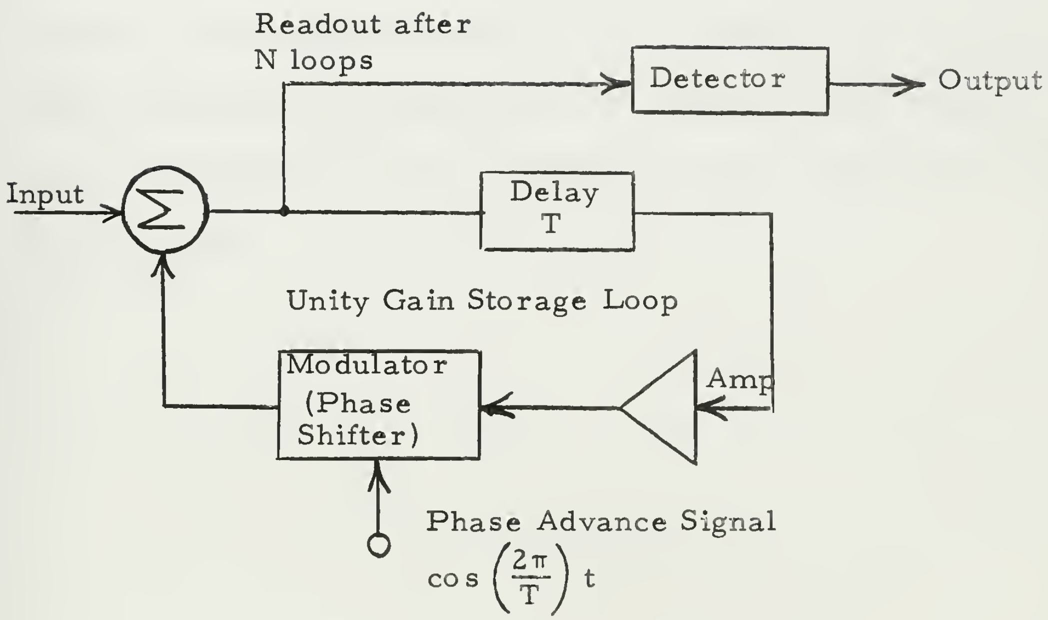


Figure 3-3. Coherent Memory Delay Integrator

The local oscillator used for the phase advance signal must be precisely matched to the delay of the delay line. Furthermore, the loop gain must be unity. With these optimum conditions assumed in being, the response of the analyzer to a single frequency is a main lobe, with side lobe patterns very much like a $(\sin x/x)^2$ curve. The side lobe patterns may

be removed in some instances by a process of selective "weighting", in which the input signal is time weighted during the analysis interval. As an example, a cosine weighting function is accomplished by amplitude modulating the input signal.

The output is produced by a linear addition of a large number of sinusoids differing from each other in frequency by small increments. Phase coincidence of all signals present in the storage loop occurs once (for a given frequency) during the observation interval. The time of this phase coincidence is linearly related to the input frequency, thus a readout of the response after the proper number of loop recirculations yields the desired periodogram. The one operational analyzer of this type with which this writer is familiar is known under the trade name "Simoramic".

4. The PPM Spectral Model

Pulse-position modulation (PPM) is a form of pulse time modulation in which the modulating process causes the position (in time) of the pulse to vary relative to its unmodulated time of occurrence. The relation of the pulse to the modulation process is normally determined by the sampling technique. The two more common types of sampling are "natural sampling and "uniform" sampling [2]. For natural sampling, the occurrence of the pulse is determined by the instant at which the leading edge of a sawtooth sampling wave and the modulating (message) voltage are equal.

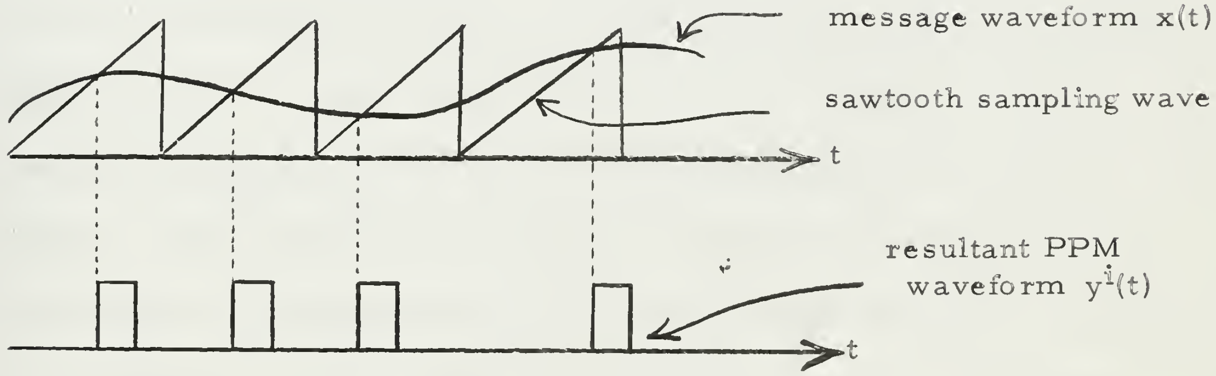


Figure 4-1. Generation of PPM by Natural Sampling.

When uniform sampling is used, the modulating waveform is sampled at equally spaced time intervals. Each pulse in the PPM waveform is then delayed in time from its unmodulated position by a time ϵ which is proportional to the sampled voltage.

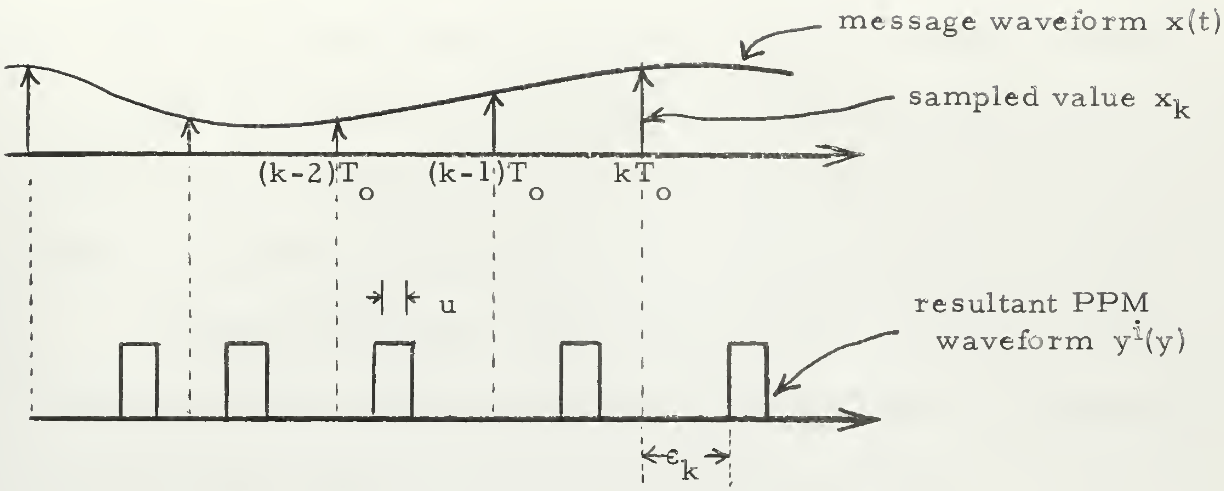


Figure 4-2. Generation of PPM by Uniform Sampling.

If one desires to relate a specific modulating process to the resultant power spectral density of the PPM waveform, it is necessary that the pulse positioning process ϵ be expressable in terms of the modulating process x. This was done for the case of uniform sampling; however, the attempt to do this for the natural sampling case proved fruitless. The resulting expression for the natural sampling case was in the form of an unwieldy series of expansion which could not be reduced to a closed form nor was of any value in its series form. Therefore, the spectral model development will be based on a single channel PPM system using uniform sampling.

The zero order hold is employed by the uniform sampling technique just described. The model can be adapted for use with other sampling schemes. For a Gaussian process, one can use a higher order hold in the model,[1] as the model will be developed to handle the case

[1] An example would be a technique which requires that the value of ϵ_k be linearly related to the average of the previous N samples of x(t).

for which $\varepsilon_m, \varepsilon_n$ are correlated Gaussian bivariate random variables for a finite m-n. The correlation for this hypothetical case can be introduced by the sampling technique, or it may be directly related to the correlation of the sampled values of the modulating process, or both.

Let the modulating waveform x(t) vary from zero to A volts. Define the maximum allowable deviation of the pulse from its unmodulated position as T_o/L seconds. The time shift ε_k is then

$$\varepsilon_k = \frac{T_o}{LA} x(kT_o) = \frac{T_o}{LA} x_k .$$

The pulse time shift ε_k is thus linearly related to the sampled amplitude of the message waveform $x(kT_o)$. The variance of ε_k is also linearly related to the variance of the message waveform samples x_k, or

$$\sigma_\varepsilon^2 = E\left\{\varepsilon - \mu_\varepsilon\right\}^2 = E\left\{\frac{T_o}{LA} x - \frac{T_o}{LA} \mu_x\right\}^2 , \text{ or}$$

$$\sigma_\varepsilon^2 = \left(\frac{T_o}{LA}\right)^2 \sigma_x^2 .$$

The standard deviation (or RMS deviation from the mean) of the pulse time shift ε_k is

$$\sigma_\varepsilon = \frac{T_o}{LA} \sigma_x$$

A more meaningful expression in determining what the actual RMS deviation is relative to a normalized interval will be defined as σ_n.

$$\sigma_n = \frac{\sigma_\varepsilon}{T_o} = \frac{\sigma_x}{LA}$$

A general expression for the power spectral density of a PPM waveform will be shown. The process x(t) producing the modulation will be considered stationary. The development of the equations will follow very closely that of Middleton [8].

A pulse function $U(t - nT_o - \epsilon)$ shall be defined as having any desired shape (with maximum value of one) and existing only for a duration u beginning at time $t = nT_o + \epsilon_n$. Outside of this specified interval, the pulse function is zero. Define y_n^i as the peak amplitude of the i^{th} member of the ensemble of PPM waveforms in the time interval $nT_o + \epsilon_n < t < nT_o + \epsilon_n + u$. A member of the truncated ensemble $y_N(t)$ may now be written:

$$y_N^i(t) = \sum_{n=-N}^{N} y_n^i U(t - nT_o - \epsilon_n^i)$$

The autocorrelation function of the PPM waveform

$$y^i(t) = \lim_{N\to\infty} y_N^i(t)$$

is $R^i(\tau)$.

$$R^i(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} y_N^i(t) y_N^i(t + \tau) dt$$

$$= \lim_{N\to\infty} \frac{1}{(2N+1)T_o} \sum_{m,n=-N}^{N} y_m^i y_n^i$$

$$\int_{-\infty}^{\infty} U(t - mT_o - \epsilon_m^i) U(t + \tau - nT_o - \epsilon_n^i) dt$$

Now let $X_U(f)$ be the Fourier transform of U(t).

$$X_U(f) = \int_{-\infty}^{\infty} U(t)e^{-j\omega t}dt$$

or

$$\int_{-\infty}^{\infty} U(t - \alpha)e^{-j\omega t}dt = X_U(f)e^{-j\omega\alpha} .$$

Also

$$U(t) = \int_{-\infty}^{\infty} X_U(f)e^{j\omega t}df$$

or

$$U(t - \beta) = \int_{-\infty}^{\infty} X_U(f)e^{j\omega(t-\beta)}df$$

Letting $\alpha = nT_o + \epsilon_n - \tau$ and $\beta = mT_o + \epsilon_m$ in the above equations, we have

$$R^i(\tau) = \left[\lim_{N\to\infty} \frac{1}{(2N + 1)T_o} \sum_{m,n=-N}^{N} y_n^i y_m^i \int_{-\infty}^{\infty} U(t - \alpha)\right.$$

$$\left.\int_{-\infty}^{\infty} X_U(f)e^{j\omega(t-\beta)}dfdt\right]$$

$$= \left[\lim_{N\to\infty} \frac{1}{(2N + 1)T_o} \sum_{m,n=-N}^{N} y_n^i y_m^i \int_{-\infty}^{\infty} X_U(f)e^{-j\omega\beta}\right.$$

$$\left.\int_{-\infty}^{\infty} U(t - \alpha)e^{j\omega t}dtdf\right]$$

$$= \left[\lim_{N\to\infty} \frac{1}{(2N + 1)T_o} \sum_{m,n=-N}^{N} y_m^i y_n^i \int_{-\infty}^{\infty} X_U(f)e^{-j\omega\beta}\right.$$

$$\left.X_U(f)^* e^{j\omega\alpha}df\right]$$

$$R^i(\tau) = \frac{1}{T_o}\int_{-\infty}^{\infty} |X_U(f)|^2 e^{-j\omega\tau}\left[\lim_{N\to\infty}\frac{1}{(2N+1)}\sum_{m,n=-N}^{N} y_n^i y_m^i e^{j\omega(\varepsilon_n^i - \varepsilon_m^i)} e^{j\omega(n-m)T_o}\right] df$$

Now is it necessary to determine the covariance function from the above. Recall that the covariance function is defined as

$$K(\tau) = E\left\{R^i(\tau)\right\}$$

Additionally, recall that it is necessary to perform the ensemble average prior to taking the time average of the autocorrelation function. Thus

$$K(\tau) = \frac{1}{T_o}\int_{-\infty}^{\infty} |X_U(f)|^2 e^{-j\omega\tau}\left[\lim_{N\to\infty}\frac{1}{(2N+1)} E\left\{\sum_{m,n=-N}^{N} y_m^i y_n^i e^{j\omega(\varepsilon_n^i - \varepsilon_m^i)} e^{j\omega(n-m)T_o}\right\}\right] df$$

Now observe three important facts. The expected value of a finite series is being taken, such that $y_{m,n}$ vanish for $|m|, |n| > N$. Since the process is stationary, the statistical relationship of any two random variables displaced the same distance kT_o in time is the same for all such pairs of random variables. Finally, there are $2N + 1$ such like processes in the double series. Define $k = m - n$ in the series, and the covariance function may be rewritten

$$K(\tau) = \frac{1}{T_o} \int_{-\infty}^{\infty} |X_U(f)|^2 e^{-j\omega\tau} \left[\lim_{N \to \infty} \sum_{k=-N}^{N} E\left\{ y_1 y_2 e^{j\omega(\epsilon_2 - \epsilon_1)} \right\}_{t_2 - t_1 = kT_o} e^{j\omega k T_o} df \right]$$

Now that the mean of the ensemble has been taken, one may apply the limit, and

$$K(\tau) = \int_{-\infty}^{\infty} \left[\frac{1}{T_o} |X_U(f)|^2 \sum_{k=-\infty}^{\infty} E\left\{ y_1 y_2 e^{j\omega(\epsilon_2 - \epsilon_1)} \right\}_{t_2 - t_1 = kT_o} e^{j\omega k T_o} \right] e^{-j\omega\tau} df$$

$K(\tau)$ is now in the form of an inverse Fourier transform; thus the Wiener-Khintchine Theorem for <u>two</u> sided density functions may be utilized at this point. Let $S'(f)$ be the two sided power spectral density.

$$S'(f) = \frac{1}{T_o} |X_U(f)|^2 \sum_{k=-\infty}^{\infty} E\left\{ y_1 y_2 e^{j\omega(\epsilon_2 - \epsilon_1)} \right\}_{kT_o} e^{j\omega k T_o} ,$$

$$-\infty < f < \infty$$

Later, the power spectral density shall be defined for positive frequencies only (one sided), and since $S'(f)$ is an even function, the one sided power spectral density is $S(f) = [2S'(f)]\, u_s(f)$, where $u_s(f)$ is the unit step function. The unit step function is defined as

$$u_s(f) = \begin{cases} 0 & , \ f < 0 \\ 1/2 & , \ f = 0 \\ 1 & , \ f > 0 \end{cases}$$

Define $S_U'(f)$ as the two sided spectral density for the typical sampling pulse, or

$$S_U'(f) = \frac{|X_U(f)|^2}{T_o}, \quad -\infty \leq f \leq \infty .$$

$S_U'(f)$ is a continuous even function of frequency. The one sided spectral density for the typical sampling pulse would then be

$$S_U(f) = 2S_U'(f) = \frac{2|X_U(f)|^2}{T_o}, \quad f \geq 0$$

$$= 0 \qquad\qquad , \quad f < 0 .$$

Thus, the power spectral density may be rewritten

$$S'(f) = S_U'(f) \sum_{k=-\infty}^{\infty} E\left\{ y_1 y_2 e^{j\omega(\varepsilon_2 - \varepsilon_1)} \right\}_{kT_o} e^{j\omega k T_o}, \quad -\infty \leq f \leq \infty .$$

The spectral model to be used assumes that the y_k are independent from sample to sample, and independent from the ε process. Thus, the equation that is used for the generalized spectral model is

$$S'(f) = S_U'(f) \left[\overline{y^2} + \overline{y}^2 \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} E\left\{ e^{j\omega(\varepsilon_2 - \varepsilon_1)} \right\}_{kT_o = t_2 - t_1} e^{j\omega k T_o} \right], \quad -\infty \leq f \leq \infty .$$

Observe that the model can also be used for pulse-amplitude modulation (PAM) in which the y_k represent the voltage level of the $k^{\underline{th}}$ sample of the message and the pulse is subjected to a random jitter (in time),

possibly due to equipment temperature fluctuations or varying transmission path length. The y_k would of necessity be independent random variables for the generalized spectral model just developed. The ϵ_k, however, as will be shown in the further development of <u>this</u> PPM model, can either be independent, or correlated random variables with a Gaussian distribution.

5. A Discussion of the Statistics in the Spectral Model

As seen in the development of the spectral model, certain statistics of the two processes ϵ and y, together with the pulse shape, in essence determine the power spectral density. Since independence was assumed, the statistics of the pulse amplitude y_k are quite simply determined, as all that is involved is the mean of the square of the process $(\overline{y^2})$ and the square of the mean $(\overline{y}^2)$. The statistics of the ϵ process are not as simple, especially since independence of ϵ_k will not be assumed for all cases. The statistics of the ϵ process are directly related to the message process x(t). It has already been shown that the variance and the standard deviation of the two processes are linearly related. Furthermore, the probability density functions (pdf) of the two processes are of the same form if one considers continuous distributions only (i. e., if x is Gaussian, ϵ is also Gaussian). Specifically, if $p_{x_k}(x)$ is the pdf for x, then

$$p_{\epsilon_k}(\epsilon) = \frac{\sigma_x}{\sigma_\epsilon} p_{x_k}\left(\frac{\sigma_x \epsilon}{\sigma_\epsilon}\right)$$

To show that this is true, recall that $\epsilon = (\sigma_\epsilon/\sigma_x)x$. Now

$$P[\epsilon_k < \epsilon] = P\left[x_k < \frac{\sigma_x \epsilon}{\sigma_\epsilon}\right]$$

But

$$P\left[x_k < \frac{\sigma_x \epsilon}{\sigma_\epsilon}\right] = \int_{-\infty}^{\sigma_x \epsilon/\sigma_\epsilon} p_{x_k}(x)dx$$

Since the pdf is the derivative of the distribution function, then

$$p_{\varepsilon_k}(\varepsilon) = \frac{d}{d\varepsilon}\left[\int_{-\infty}^{\sigma_x \varepsilon/\sigma_\varepsilon} p_{x_k}(x)dx\right]$$

from which one can obtain the desired result

$$p_{\varepsilon_k}(\varepsilon) = \frac{\sigma_x}{\sigma_\varepsilon}\, p_{x_k}\left(\frac{\sigma_x \varepsilon}{\sigma_\varepsilon}\right)$$

To avoid carrying unnecessary terms in equations already complex with terminology, a time axis shift of the PPM waveform will be assumed such that the mean of the process ε is zero. That this simplification may be performed with no effect on the resultant power spectra is intuitively obvious, but nevertheless has been previously verified by the writer in each case considered.

To evaluate the statistics of the process ε that appear in the equation for the power spectral density, one must determine the joint characteristic function $G(j\omega_1, j\omega_2)$ of a bivariate process. The joint characteristic function is defined as

$$G(j\omega_1, j\omega_2) = E\left\{e^{j(\omega_1\varepsilon_1 + \omega_2\varepsilon_2)}\right\} = E\left\{e^{j\omega(\varepsilon_2 - \varepsilon_1)}\right\} .$$

This joint characteristic function will be evaluated for the Gaussian bivariate case. For the remainder of the cases included in the computer model, statistical independence from interval to interval of the ε process was assumed. For the independent case, the problem reduces to the determination of the square of the absolute value of the characteristic function of ε.

$$E\left\{e^{j\omega(\epsilon_2-\epsilon_1)}\right\} = \left|E\left\{e^{j\omega\epsilon}\right\}\right|^2 = |G_\epsilon(j\omega)|^2$$

The equation for the power spectral density now reduces to

$$S'(f) = S_U'(f)\left[\overline{y^2} + \bar{y}^2\,|G_\epsilon(\omega)|^2 \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} e^{j\omega k T_o}\right] \quad -\infty \leq f \leq \infty$$

But

$$\sum_{-\infty}^{\infty} e^{j\omega k T_o} = \frac{1}{T_o}\sum_{-\infty}^{\infty} \delta\left(f - \frac{k}{T_o}\right)$$

Recall that $S_U(f) = 2S'(f)$ for $f \geq 0$ and $S(f) = 2S'(f)u_s(f)$. Then

$$S(f) = S_U(f)\left\{\overline{y^2} + \bar{y}^2\,|G_\epsilon(\omega)|^2\left[-1 + \frac{1}{T_o}\sum_{k=0}^{\infty}\delta\left(f - \frac{k}{T_o}\right)\right]\right\} u_s(f)$$

for the case in which ϵ_k and y_k are independent from sampling interval to sampling interval. As mentioned previously, only positive frequencies are considered.

5A. The Message as a Gaussian Process

The purpose of this subsection (and the following subsections) is to fit a specific process into the generalized spectral model. The process considered may either be the message process x or the ϵ process, as the similarity of the two processes has been previously shown. For the Gaussian case, the development will allow for a statistical correlation to exist between I adjacent samples of the process. More specifically, this development will allow for the spectral model to handle Gaussian processes for which the autocorrelation function $R(\tau)$ exists for $x(t)$ up to $|\tau| = IT_o$, and $R(\tau) = 0$ for $|\tau| > IT_o$. This is practical because a purely random process yields an autocorrelation function that approaches zero as τ increases [1].

The expected value

$$E\left\{e^{j\omega(\epsilon_2-\epsilon_1)}\right\}_{kT_o}$$

is the joint characteristic function of ϵ_2, ϵ_1, for a given $kT_o = t_2 - t_1$. The joint characteristic function $G(j\omega_1, j\omega_2)$ as previously defined fits the case now under development when $\omega_1 = -\omega_2$. It was decided to carry out the general development of $G(j\omega_1, j\omega_2)$, substituting into the final result $\omega = \omega_2 = -\omega_1$. Thus,

$$E\left\{e^{j\omega(\epsilon_2-\epsilon_1)}\right\} = E\left\{e^{j(\omega_1\epsilon_1+\omega_2\epsilon_2)}\right\} \equiv G(j\omega_1, j\omega_2)$$

The expected value is determined as

$$G(j\omega_1, j\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{(j\omega_1\epsilon_1 + j\omega_2\epsilon_2)} p(\epsilon_1, \epsilon_2) d\epsilon_1 d\epsilon_2$$

The bivariate Gaussian pdf is

$$p(\epsilon_1, \epsilon_2) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} e^{[-1/2\sigma^2(1-\rho^2)](\epsilon_1{}^2 - 2\rho\epsilon_1\epsilon_2 + \epsilon_2{}^2)}$$

where

$$\mu_\epsilon = 0 \text{ and } \sigma_{\epsilon_1} = \sigma_{\epsilon_2} = \sigma$$

For simplicity, substitute

$$\epsilon_1 = \lambda \qquad \epsilon_2 = \psi$$

$$2\sigma^2 = a \qquad \sqrt{1-\rho^2} = b$$

Then

$$G(j\omega_1 j\omega_2) = \frac{1}{\pi ab}\int_{-\infty}^{\infty} e^{j\omega_2\psi - (\psi^2/ab^2)} \int_{-\infty}^{\infty} e^{-(1/ab^2)(\lambda^2 - 2\rho\lambda\psi)} e^{j\omega_1\lambda} d\lambda\, d\psi$$

But

$$\int_{-\infty}^{\infty} e^{-(1/ab^2)(\lambda^2 - 2\rho\lambda\psi)} e^{j\omega_1\lambda} d\lambda$$

$$= e^{\rho^2\psi^2/ab^2}\int_{-\infty}^{\infty} e^{-(1/ab^2)[(\lambda-\rho\psi)^2]} e^{j\omega_1\lambda} d\lambda$$

which is in the form of a constant multiplying the characteristic function of a single Gaussian process λ, with mean $\mu_\lambda = \rho\psi$ and $\sigma_\lambda{}^2 = ab^2/2$. The characteristic function of a single Gaussian process may easily be

determined (with the aid of tables of integrals), and the last integral is simply

$$b\sqrt{a\pi}\ e^{\rho^2\psi^2/ab^2}\left[e^{j\rho\psi\omega_1-(ab^2\omega_1^2/4)}\right]$$

The joint characteristic function then becomes

$$G(j\omega_1 j\omega_2) = \frac{1}{\sqrt{a\pi}}\ e^{-(ab^2\omega_1^2/4)} \int_{-\infty}^{\infty} e^{(\rho^2-1)/(ab^2)\psi^2}\ e^{j(\omega_2+\rho\omega_1)\psi}\ d\psi\ .$$

Define

$$\omega_3 = \omega_2 + \rho\omega_1$$

$$\sigma_\psi^2 = -\frac{1}{2}\left(\frac{ab}{\rho^2 - 1}\right)$$

and this integral also takes the form of the single characteristic function of the process ψ, with the mean $\mu_\psi = 0$

$$G(j\omega_1, j\omega_2) = \sigma_\psi\sqrt{\frac{2}{a}}\ e^{-(ab^2\omega_1^2)/4}\left[\frac{1}{\sigma_\psi\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(1/2\sigma_\psi^2)\psi^2}\ e^{j\omega_3\psi}\ d\psi\right]$$

$$= \sigma_\psi\sqrt{\frac{2}{a}}\ e^{-(ab^2\omega_1^2/4)}\left[e^{-(\sigma_\psi^2\omega_3^2/2)}\right]$$

By resubstituting the values for σ_ψ, a, b, and ω_3 one obtains

$$G(j\omega_1, j\omega_2) = e^{-(\sigma_\epsilon^2/2)(\omega_1^2+2\rho\omega_1\omega_2+\omega_2^2)}$$

Recall that for this development, $\omega = \omega_2 = -\omega_1$. Thus,

$$G(j\omega_1 j\omega_2) = G(\omega) = e^{-\sigma_\epsilon^2\omega^2(1-\rho)}$$

Applying the above to the spectral model for a given value of kT_o, we

have

$$E\left\{e^{j\omega(\epsilon_2-\epsilon_1)}\right\}_{kT_o=t_2-t_1} = e^{-\sigma_\epsilon^2\omega^2[1-\rho(kT_o)]}$$

That the expression is a function of the autocorrelation of the process ϵ (or x) may be determined by observing that ρ is the correlation coefficient for a given kT_o in the development. Considering all kT_o, $\rho(kT_o)$ is the normalized autocorrelation function.

$$\rho(\tau) = \frac{R(\tau)}{R(0)}$$

Now let $R(\tau) = 0$ for $k > I$. The power spectral density for this case is then

$$S'(f) = S_U'(f)\left\{\overline{y^2} + \overline{y}^2\left[\sum_{\substack{k=-I\\k\neq 0}}^{I} e^{-\sigma_\epsilon^2\omega^2[1-\rho(k)]}\, e^{j\omega kT_o} + e^{-\sigma_\epsilon^2\omega^2}\left(\sum_{k=-\infty}^{-(I+1)} e^{j\omega kT_o} + \sum_{k=(I+1)}^{\infty} e^{j\omega kT_o}\right)\right]\right\}$$

But $\rho(kT_o) = \rho(-kT_o)$, as $R(\tau)$ is an even function, and

$$S'(f) = S_U'(f)\left\{\overline{y^2} + \overline{y}^2\left[2\sum_{k=1}^{I}\left(e^{-\sigma_\epsilon^2\omega^2[1-\rho(k)]} - e^{-\sigma_\epsilon^2\omega^2}\right)\cos\omega kT_o + e^{-\sigma_\epsilon^2\omega^2}\left(-1 + \sum_{k=-\infty}^{\infty} e^{j\omega kT_o}\right)\right]\right\}$$

and the spectral model for the Gaussian case (for $f \geq 0$) becomes

$$S(f) = S_U(f)\left\{\overline{y^2} + \overline{y}^2\left[2\sum_{k=1}^{I}\left(e^{-\sigma_\epsilon^2\omega^2[1-\rho(k)]} - e^{-\sigma_\epsilon^2\omega^2}\right)\cos\omega kT_o + e^{-\sigma_\epsilon^2\omega^2}\left\{-1 + \frac{1}{T_o}\sum_{k=0}^{\infty}\delta\left(f - \frac{k}{T_o}\right)\right\}\right]\right\} u_s(f)$$

5B. The Message as a Uniform Process

The specific spectral model will be developed for the case in which the message determining the modulation of the PPM waveform is a uniform process. Statistical independence of both the ε and the y processes will be assumed. For this assumption, the spectral density was shown to be

$$S(f) = S_U(f)\left\{\overline{y^2} + \overline{y}^2 \left|G_\varepsilon(\omega)\right|^2 \left[-1 + \frac{1}{T_o}\sum_{k=0}^{\infty} \delta\left(f - \frac{k}{T_o}\right)\right]\right\} u_s(f)$$

The problem is therefore reduced to that of determining the characteristic function $G_\varepsilon(\omega)$ of a uniform process.

$$G_\varepsilon(\omega) = \int_{-\infty}^{\infty} e^{j\omega\varepsilon} p_{\varepsilon_k}(\varepsilon) d\varepsilon$$

As shown previously, if x is uniformly distributed, then ε is also uniformly distributed. For the case of uniform distribution of x, the total possible deviation ε(max) allowed is T_o/L seconds (or $\pm(T_o/2L)$ from the zero mean assumed). Therefore,

$$p_{\varepsilon_k}(\varepsilon) = \frac{L}{T_o}, \quad -\frac{T_o}{2L} \le \varepsilon \le \frac{T_o}{2L}$$

Thus,

$$G_\varepsilon(\omega) = \int_{-(T_o/2L)}^{T_o/2L} \frac{L}{T_o} e^{j\omega\varepsilon} d\varepsilon$$

or

$$G_\varepsilon(\omega) = \frac{\sin \frac{\omega T_o}{2L}}{\frac{\omega T_o}{2L}}$$

The power spectral density for the uniform case then becomes

$$S(f) = S_U(f)\left\{\overline{y^2} + \bar{y}^2\left(\frac{\sin\frac{\omega T_o}{2L}}{\frac{\omega T_o}{2L}}\right)^2\left[-1 + \frac{1}{T_o}\sum_{k=0}^{\infty}\delta\left(f - \frac{k}{T_o}\right)\right]\right\} u_s(f)$$

5C. The Message as an Exponential Process

The specific spectral model will be developed for the case in which the message determining the modulation of the PPM waveform is an exponential process. Of interest for this case is that the speech process is approximately exponentially distributed, especially in the region of medium to large amplitudes [1].

For the exponential case, statistical independence will again be assumed for both the ε and y processes. As was the case then in section 5B, the problem is reduced to determining the characteristic function of the process ε.

The exponential probability density function is

$$p_{\varepsilon_k}(\varepsilon) = \frac{1}{\sqrt{2}\,\sigma_\varepsilon} e^{-(\sqrt{2}/\sigma_\varepsilon)|\varepsilon|}$$

from which one can determine

$$\mu_\varepsilon = E(\varepsilon) = 0$$

$$\mathrm{Var}(\varepsilon) = E(\varepsilon^2) - \mu_\varepsilon^{\;2} = E(\varepsilon^2) = \sigma_\varepsilon^{\;2}$$

Define

$$\alpha \equiv \frac{\sqrt{2}}{\sigma_\varepsilon}$$

Then

$$p_{\varepsilon_k}(\varepsilon) = \frac{\alpha}{2} e^{-\alpha|\varepsilon|}$$

The characteristic function is

$$G_\varepsilon(\omega) = \int_{-\infty}^{\infty} e^{j\omega\varepsilon} p_{\varepsilon_k}(\varepsilon)d\varepsilon$$

$$= \frac{\alpha}{2}\left[\int_{-\infty}^{0} e^{\varepsilon(j\omega+\alpha)}d\varepsilon + \int_{0}^{\infty} e^{\varepsilon(j\omega-\alpha)}d\varepsilon\right]$$

or

$$G_\varepsilon(\omega) = \frac{\alpha^2}{\omega^2 + \alpha^2}$$

The power spectral density of the PPM waveform for the exponential case is then

$$S(f) = S_U(f)\left\{\overline{y^2} + \overline{y}^2 \frac{\alpha^4}{(\omega^2 + \alpha^2)^2}\left[-1 + \frac{1}{T_o}\sum \delta\left(f - \frac{k}{T_o}\right)\right]\right\} u_s(f)$$

6. The Computer Model

The PPM spectral model was developed in section four, and the model was adapted to specific modulation processes in section five. The resultant equations are quite formidable if manual calculation is required to determine the power spectral density for various cases. Fortunately, the tedium of manual calculations of such proportions is not necessary nowadays, as the digital computer handles such tasks with great facility. A computer model will now be developed that will incorporate all cases previously discussed, plus the unmodulated case.

The computer model developed relates the pulse positioning process ε in terms of the original modulating process. This allows the user to have a more direct insight as to the effects of the modulating process on the power spectral density of the PPM waveform. Most of the parameters of the model are variable at the discretion of the user. An exception is the number of points of continuous spectra calculated between each possible discrete frequency spectrum. Since five points in each such interval are adequate for a graphic plot, this quantity has been fixed at five to avoid possible confusion by future users of the model.

It should be noted here that the following development of the computer model may not at times appear to be the simplest approach. The approach taken generally yields simplification of computer Fortran operations to be performed, such as eliminating the calling of a library function for performing an exponentation when a simple multiplying

operation will suffice. It is admitted, however, that a few of the "simplifying" techniques employed did not necessarily make the program more efficient. The development will include these also, as they have become a part of the final computer model.

While no extreme efforts were applied toward making the computer program efficient, this aspect was certainly kept in mind while the program was being developed.

The program was written in Fortran IV language for the IBM 7040 at Sylvania's Electronic Defense Laboratories (EDL). With some obvious card changes (for each library function used), the program can be adapted to the Fortran language of the CDC 1604 at the Naval Postgraduate School.

The total run time (including card reading and program printout) for computing four separate cases in one program run from a binary deck was timed at two minutes on the 7040.

From the previous developments, it was decided to base the computer model on the mathematical model expressed in the following form for frequencies greater than zero. The case for d-c (zero frequency) will be developed separately.

$$S(f) = S_U(f)\left\{\overline{y^2} + \overline{y}^2[\mathrm{Sum}(f) - |G_\varepsilon(\omega)|^2]\right\}$$
$$+ S_U(f)\left\{\overline{y}^2 |G_\varepsilon(\omega)|^2 \frac{1}{T_o} \sum_{k=1}^{\infty} \delta\left(f - \frac{k}{T_o}\right)\right\}$$

The first half of the expression is the continuous spectrum and the

second half is the spectrum concentrated at discrete frequencies. The term Sum(f) is defined as

$$\mathrm{Sum}(f) \triangleq 2\sum_{k=1}^{I}\left(e^{-(\sigma_\epsilon \omega)^2[1-\rho(k)]} - e^{-(\sigma_\epsilon \omega)^2}\right)\cos k\omega T_o .$$

The term Sum(f) is included in the model so that the Gaussian correlated process may be handled. Note that when Sum(f) equals zero, the model reduces to that derived for the case in which all processes were assumed independent. The term $G_\epsilon(\omega)$ is simply the characteristic function of the independent random process ϵ, and fits as shown into the chosen representation even when the Gaussian correlated case is considered. That this is true may be determined from the fact that the characteristic function for the independent Gaussian case (mean zero) is the square root of that determined for the Gaussian bivariate case when rho (ρ) is zero, or

$$G_\epsilon(\omega) = e^{-[(\sigma_\epsilon \omega)^2/2]}$$

(for Gaussian process with $\rho = 0$).

All terms in the model have now been adequately defined except for the power spectral density of the "typical sampling pulse". Recall in a previous development that this pulse could be of any shape with a maximum amplitude of one. Also recall that this amplitude of one does not restrict the amplitude of the pulses in the PPM waveform, as the PPM pulse amplitudes are determined by the y process. Thus, the "typical sampling pulse" is simply a description of the <u>shape</u> of the

pulses of the PPM pulse train. For this computer model, a rectangular pulse will be assumed, the pulse being of duration u and of unit amplitude.

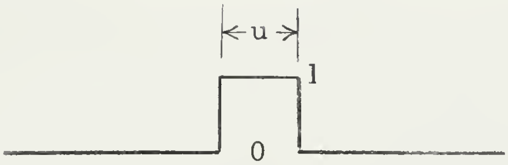


Figure 6-1. The Typical Sampling Pulse for the Computer Model.

As previously shown,

$$S_U(f) = \frac{2}{T_o} |X_U(f)|^2 .$$

Here

$$X_U(f) = \int_{-\infty}^{\infty} U(t)e^{-j\omega t}dt = \int_{-(u/2)}^{u/2} (1)e^{-j\omega t}dt .$$

from which one may determine

$$S_U(f) = \frac{2u^2}{T_o} \left(\frac{\sin \frac{\omega u}{2}}{\frac{\omega u}{2}} \right)^2 .$$

To simplify the computer program (so that some of the time consuming operations are minimized), the above equation may be rewritten as

$$S_U(f) = \frac{4u^2}{T_o} \frac{(1 - \cos \omega u)}{(\omega u)^2}$$

The discrete frequencies at which S(f) is to be computed are

$$f_n = \frac{n}{5T_o}, \text{ or } \omega_n = \frac{2\pi n}{5T_o}, \quad 0 \le n \le 200$$

To calculate S(0), one must apply the value of the unit step function (one-half) at zero frequency (the d-c term). Substituting $\omega = 0$ into the equations developed thus far, one obtains for all cases

$$S(0) = \frac{S_U(0)}{2} \left\{ \overline{y^2} - \overline{y}^2 \left(1 - \frac{1}{T_o} \right) \right\}$$

where

$$\frac{S_U(0)}{2} = \frac{u^2}{T_o}$$

The sequence of computation of the power spectral density may be determined from the flow chart (Figure 6-4).

The computer model will determine two hundred values of S(f) up to and including $f = 40/T_o$. This includes all 40 possible values of S(f) concentrated at discrete frequencies (delta functions) within the frequency range considered.

A list of the significant symbols used in the program and their relationship to the terms in the mathematical model is shown in Table 6-1. The computer program itself with one page of output is included as Figures 6-2 and 6-3.

Table 6-1. Computer Program Symbol Identification.

Symbol	Equivalent Symbol in Mathematical Development or Symbol Definition
A	A
ALFA2	α^2 (used for exponential case)
CFSQ(N)	$\|G_\epsilon(\omega)\|^2$
CONT(N)	Value of continuous spectrum at $f = N/5T_o$
DELFN	Value of discrete spectrum component
FL	L
F(N)	Frequency; $F(N) = N/5T_o$ cps
I	I (upper summation index for Gaussian case)
J	None; this index specifies modulation process. See comments in Figure 6-2.
NR	Number of separate cases to be computed this program run
RHO(K)	$\rho(k)$
SIGMA	σ_x
S(N)	S(f); Power spectral density
SU(N)	$S_u(f)$
SUM(N)	Summation for Gaussian correlated case
T	T_o
TEMP(K)	Partial value of summation of SUM(N)
U	U
YBSQ	$\bar{y}^2$
YSQB	$\overline{y^2}$

```
C      PROGRAM PPM
C      THIS PROGRAM CALCULATES THE POWER SPECTRAL DENSITY OF A PPM
C      WAVEFORM THAT HAS AS ITS MODULATING SOURCE ONE OF THE FOLLOWING
C      EITHER AN INDEPENDENT GAUSSIAN PROCESS OR A GAUSSIAN PROCESS
C      WITH UP TO TEN CORRELATION VALUES(FOR TAU=T,2T,...,10T), AN
C      INDEPENDENT UNIFORM PROCESS, OR AN INDEPENDENT EXPONENTIAL PROCESS
C      J=1 FOR GAUSSIAN,=2 FOR UNIFORM,=3(UNMODULATED),=4 FOR EXPONENTIAL
C      FORMAT 8 STATEMENT ALLOWS FOR READING RECORD IN FOLLOWING ORDER
C      U,T,YBSQ,YSQB,SIGMA,A,FL,J,I,AND 10 VALUES OF RHO
       DIMENSION RHO(10),SUM(200),CFSQ(200),SU(200),S(200),TEMP(10)   ,
      1F(200), CONT(200)
     6 FORMAT (4H F= F10.3, 6H S(F)= E12.4, 9H CONT(F)= E12.4)
     7 FORMAT (/5X,1HU,11X,1HT,9X,4HYBSQ,8X,4HYSQB,6X,5HSIGMA,6X,1HA,10X,
      12HFL,4X,29H 2ND ROW IS J,I,  RHO(K=1,10)/)
     8 FORMAT (7F11.6/(2I3,10F6.3))
     9 FORMAT (4H F= F10.3, 6H S(F)= E12.4)
       PI= 3.14159265
       NR = 4
   300 PRINT 7
       READ 8, U,T,YBSQ,YSQB,SIGMA,A,FL,J,I,(RHO(K),K=1,10)
       PRINT 8, U,T, YBSQ,YSQB,SIGMA,A,FL,J,I,(RHO(K),K=1,10)
       DO 301 N=1,200
   301 SUM(N) = 0.0
       GO TO (1,2,3,4),J
     1 C= (0.4*PI*SIGMA/(FL*A))**2
       DO 11 N=1,200
       FLO2 = N
       CC=C*FLO2*FLO2
       IF (CC.LT.23.0) GO TO 12
       CFSQ(N) = 0.0
       GO TO 11
    12 CFSQ(N) = 1.0/(EXP(CC))
    11 CONTINUE
       IF (I.EQ. 0) GO TO 100
       DO 18 N=1,200
       FLO3 = N
       TEM = 0.0
       DO 15 K=1,I
       FLO4 = K
       CD = C*FLO3*FLO3*(1.0-RHO(K))
       IF (CD.LT.23.0) GO TO 14
       TEMP(K) = 0.0
       GO TO 15
    14 TEMP(K) = (1.0/(EXP(CD))-CFSQ(N))*(COS(0.4*PI*FLO3*FLO4))
    15 TEM = TEM+TEMP(K)
    18 SUM(N) = 2.0*TEM
       GO TO 100
     2 PU = 0.2*PI/FL
       DO 25 N=1,200
       FLO6=N
       PUN=PU*FLO6
    25 CFSQ(N)=((SIN(PUN))/PUN)** 2
       GO TO 100
     3 DO 35 N=1,200
    35 CFSQ(N) = 1.0
```

Figure 6-2. Source Program for PROGRAM PPM, Page 1

```
      GO TO 100
    4 ALFA2 = 2.0*(FL*A/(T*SIGMA))**2
      PE = (0.4*PI/T)** 2
      DO 45 N=1,200
      FL05 = N
   45 CFSQ(N)=(ALFA2/(PE*FL05*FL05 +ALFA2))** 2
  100 AA = 4.0*U*U/T
      B= 0.4*PI*U/T
      DF= 0.2/T
      DO 110 N=1,200
      FL01 =N
      F(N)=FL01 * DF
      G = B*FL01
  110 SU(N) = AA*(1.0-COS(G))/(G*G)
      SUZERO = AA/4.0
      SZERO = SUZERO*(YSQB -YBSQ*(1.0-1.0/T))
      FZERO= 0.0
      DO 120 N=1,200
  120 S(N) = SU(N)*(YSQB+YBSQ*(SUM(N)-CFSQ(N)))
      DO 125 N=5,200,5
      CONT(N) = S(N)
      DELFN = SU(N)*YBSQ*CFSQ(N)*1.0/T
  125 S(N) = CONT(N)+DELFN
      PRINT 9, FZERO, SZERO
      DO 210 K=1,196,5
      K3=K+3
      DO 200 N=K,K3
  200 PRINT 9, F(N),S(N)
      N=K+4
  210 PRINT 6,F(N),S(N),CONT(N)
      NR = NR-1
      IF (NR.GT.0) GO TO 300
C     THIS RUN FOR 10 KC SAMPLING.  U = 0.1T
      STOP
      END
```

Figure 6-2, Page 2

```
     U            T            YBSQ         YSQB       SIGMA        A            FL      2ND ROW IS J,I,  RHO(K=1,10)

   0.000195    0.001000    2.720000    2.720000    2.800000   10.000000    2.500000
  1  0 0.    -0.    -0.    -0.    -0.    -0.    -0.    -0.    -0.    -0.
  F=        0.    S(F)=  0.1034E-00
  F=    200.000   S(F)=  0.4037E-05
  F=    400.000   S(F)=  0.1544E-04
  F=    600.000   S(F)=  0.3228E-04
  F=    800.000   S(F)=  0.5183E-04
  F=   1000.000   S(F)=  0.1111E-00  CONT(F)=   0.7118E-04
  F=   1200.000   S(F)=  0.8779E-04
  F=   1400.000   S(F)=  0.9992E-04
  F=   1600.000   S(F)=  0.1067E-03
  F=   1800.000   S(F)=  0.1083E-03
  F=   2000.000   S(F)=  0.1693E-01  CONT(F)=   0.1052E-03
  F=   2200.000   S(F)=  0.9845E-04
  F=   2400.000   S(F)=  0.8926E-04
  F=   2600.000   S(F)=  0.7863E-04
  F=   2800.000   S(F)=  0.6743E-04
  F=   3000.000   S(F)=  0.7171E-03  CONT(F)=   0.5632E-04
  F=   3200.000   S(F)=  0.4577E-04
  F=   3400.000   S(F)=  0.3611E-04
  F=   3600.000   S(F)=  0.2754E-04
  F=   3800.000   S(F)=  0.2015E-04
  F=   4000.000   S(F)=  0.1906E-04  CONT(F)=   0.1399E-04
  F=   4200.000   S(F)=  0.9059E-05
  F=   4400.000   S(F)=  0.5300E-05
  F=   4600.000   S(F)=  0.2634E-05
  F=   4800.000   S(F)=  0.9541E-06
  F=   5000.000   S(F)=  0.1363E-06  CONT(F)=   0.1357E-06
  F=   5200.000   S(F)=  0.3941E-07
  F=   5400.000   S(F)=  0.5192E-06
  F=   5600.000   S(F)=  0.1428E-05
  F=   5800.000   S(F)=  0.2622E-05
  F=   6000.000   S(F)=  0.3967E-05  CONT(F)=   0.3967E-05
  F=   6200.000   S(F)=  0.5343E-05
  F=   6400.000   S(F)=  0.6644E-05
  F=   6600.000   S(F)=  0.7784E-05
  F=   6800.000   S(F)=  0.8699E-05
  F=   7000.000   S(F)=  0.9344E-05  CONT(F)=   0.9344E-05
  F=   7200.000   S(F)=  0.9694E-05
  F=   7400.000   S(F)=  0.9746E-05
  F=   7600.000   S(F)=  0.9512E-05
  F=   7800.000   S(F)=  0.9020E-05
  F=   8000.000   S(F)=  0.8310E-05  CONT(F)=   0.8310E-05
  F=   8200.000   S(F)=  0.7430E-05
  F=   8400.000   S(F)=  0.6433E-05
```

Figure 6-3. A Sample Output from PROGRAM PPM

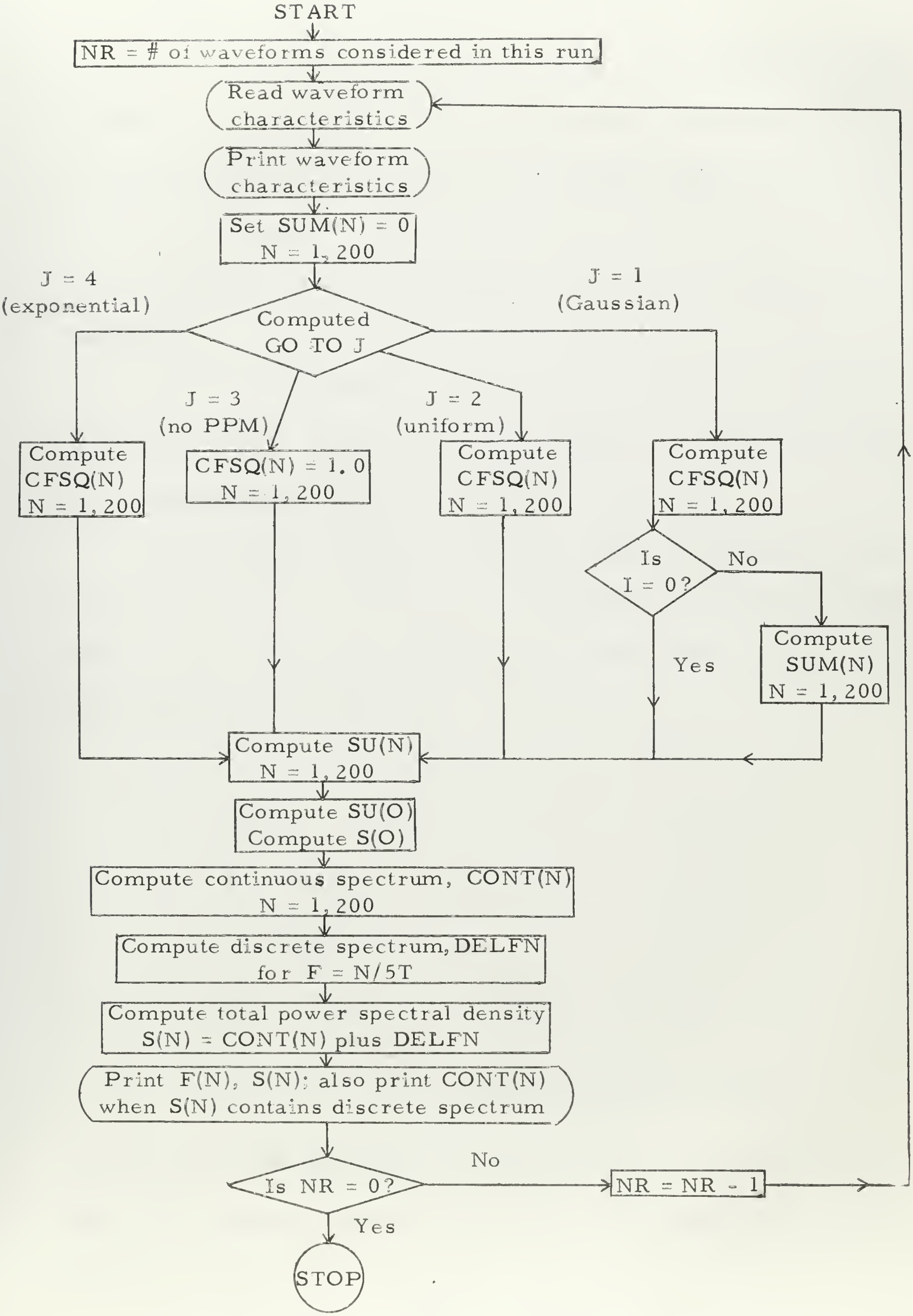


Figure 6-4. Flow Chart for PROGRAM PPM.

7. Computed Power Spectral Densities, Results

This section contains the displays of the power spectral densities computed for certain combinations of pulse widths, modulation processes, and varying degrees of pulse deviation (or RMS deviation from the mean position). The legend accompanying each graph indicates the pertinent data regarding the process for which that power spectral density was computed.

The first graph (Figure 7-4) is the power spectrum of an unmodulated pulse train. Observe that with no modulation of the pulse train, there is no continuous component in the power spectrum.

Three different normalized autocorrelation functions were used for the Gaussian case. For the process which has very little correlation between samples, the approximation used was the linear autocorrelation function indicated by Figure 7-1.

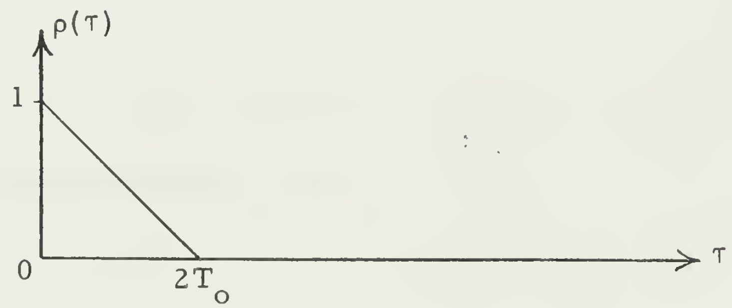


Figure 7-1. Limited Correlation Between Samples.

An approximation to the autocorrelation function of a process with a high degree of correlation existing for several adjacent samples is shown in Figure 7-2.

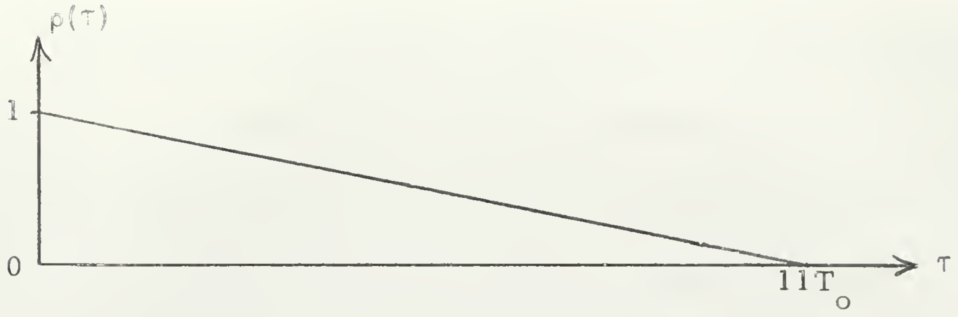


Figure. 7-2. Samples Highly Correlated.

An approximation to the autocorrelation existing for band-limited Gaussian noise (0-W cps) is shown in Figure 7-3. In Figure 7-3, the Gaussian noise is assumed to have been sampled at ten times the highest frequency present, or at 10W samples per second.

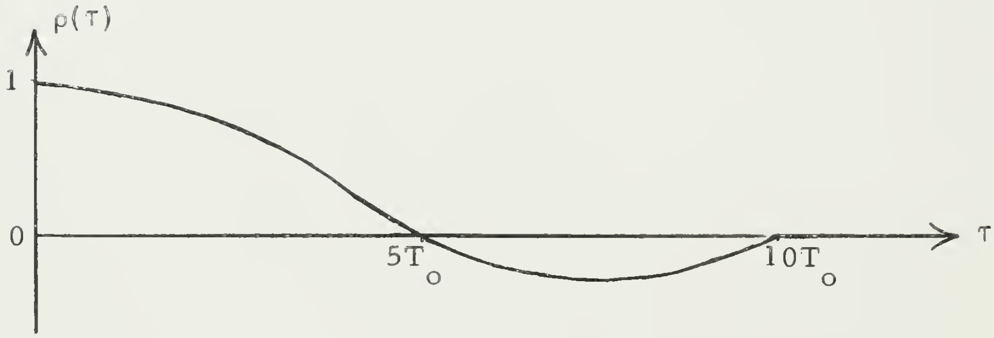


Figure 7-3. Approximate Autocorrelation for Oversampled Band-limited Gaussian Noise.

Two examples are shown for which the pulse amplitudes are not held constant (Rayleigh distributed), in addition to being pulse position modulated. It may be observed that in these cases the continuous spectrum begins with an appreciable value at zero frequency, which is a characteristic of pulse amplitude modulated waveforms.

To allow for an adequate display of the continuous spectrum, a scale factor was chosen that does not permit the showing of the discrete power spectrum to scale. The values of the discrete power spectral lines are numerically indicated for each line that exceeds the scale of the graph.

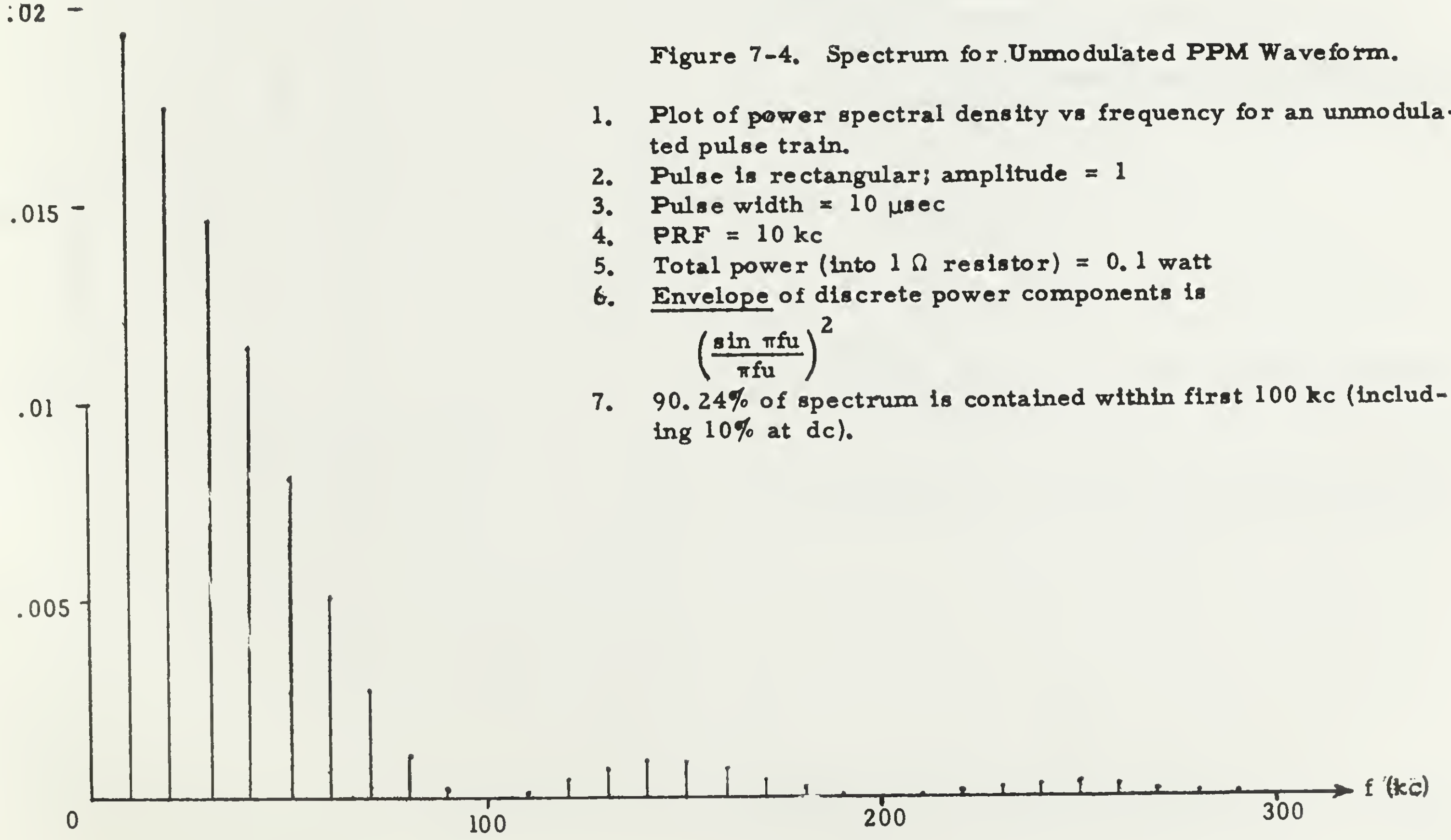


Figure 7-4. Spectrum for Unmodulated PPM Waveform.

1. Plot of power spectral density vs frequency for an unmodulated pulse train.
2. Pulse is rectangular; amplitude = 1
3. Pulse width = 10 μsec
4. PRF = 10 kc
5. Total power (into 1 Ω resistor) = 0.1 watt
6. <u>Envelope</u> of discrete power components is

$$\left(\frac{\sin \pi f u}{\pi f u}\right)^2$$

7. 90.24% of spectrum is contained within first 100 kc (including 10% at dc).

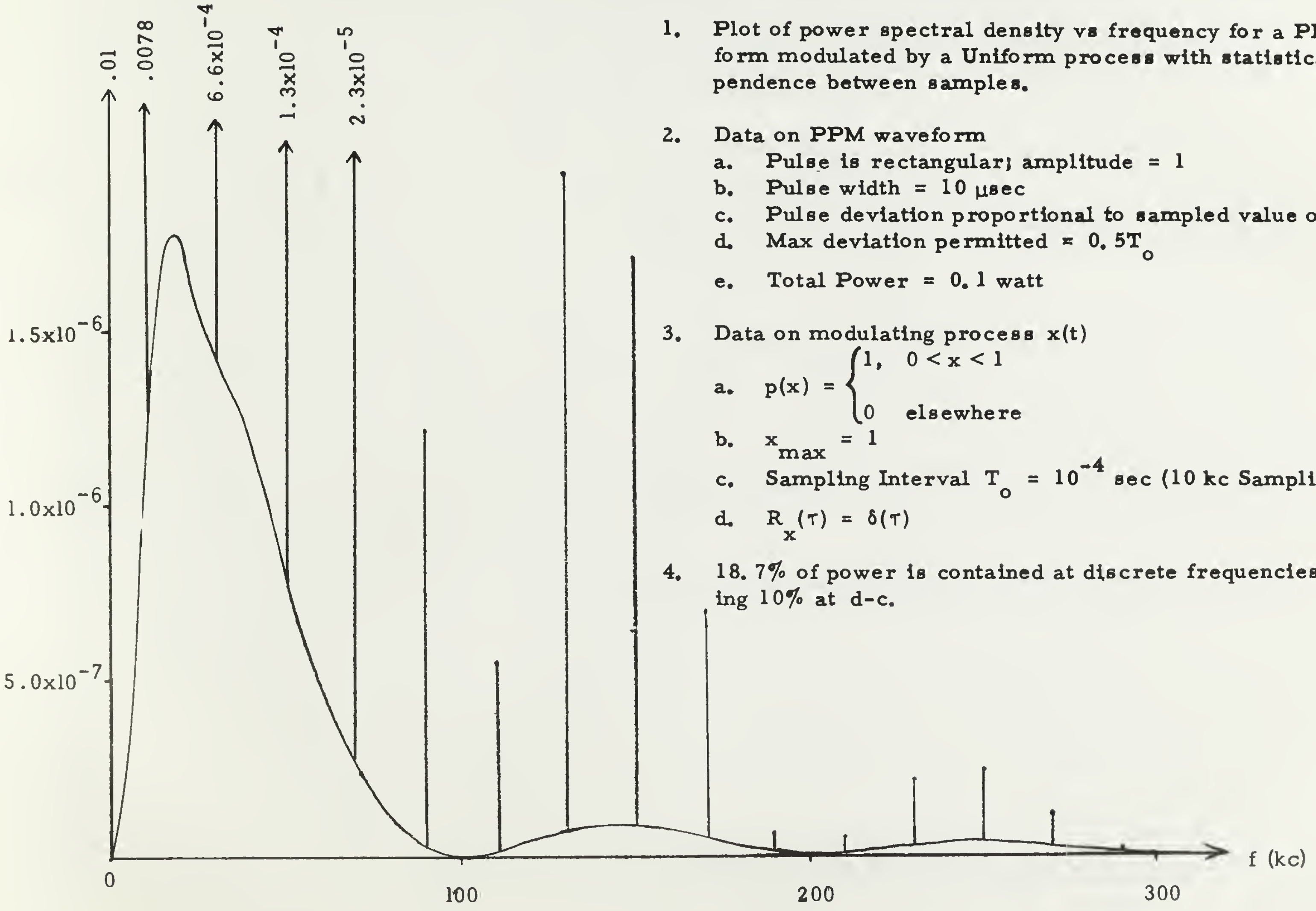


Figure 7-5. Spectrum for Uniform Case #1.

1. Plot of power spectral density vs frequency for a PPM waveform modulated by a Uniform process with statistical independence between samples.

2. Data on PPM waveform
   a. Pulse is rectangular; amplitude = 1
   b. Pulse width = 10 μsec
   c. Pulse deviation proportional to sampled value of x(t).
   d. Max deviation permitted = $0.5T_o$
   e. Total Power = 0.1 watt

3. Data on modulating process x(t)
   a. $p(x) = \begin{cases} 1, & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$
   b. $x_{max} = 1$
   c. Sampling Interval $T_o = 10^{-4}$ sec (10 kc Sampling Rate)
   d. $R_x(\tau) = \delta(\tau)$

4. 18.7% of power is contained at discrete frequencies, including 10% at d-c.

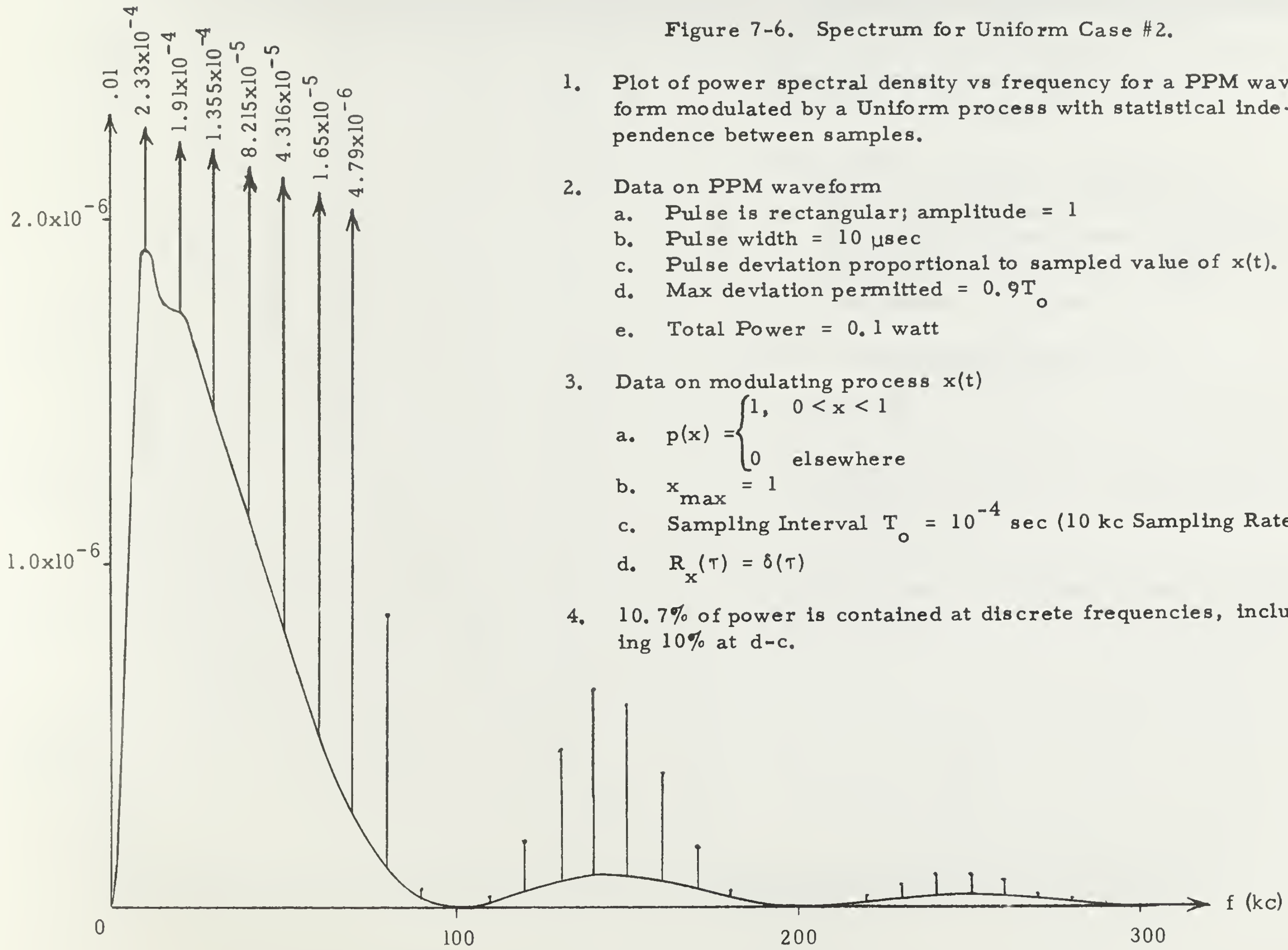


Figure 7-6. Spectrum for Uniform Case #2.

1. Plot of power spectral density vs frequency for a PPM waveform modulated by a Uniform process with statistical independence between samples.

2. Data on PPM waveform
   a. Pulse is rectangular; amplitude = 1
   b. Pulse width = 10 μsec
   c. Pulse deviation proportional to sampled value of $x(t)$.
   d. Max deviation permitted = $0.9T_o$
   e. Total Power = 0.1 watt

3. Data on modulating process $x(t)$
   a. $p(x) = \begin{cases} 1, & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$
   b. $x_{max} = 1$
   c. Sampling Interval $T_o = 10^{-4}$ sec (10 kc Sampling Rate)
   d. $R_x(\tau) = \delta(\tau)$

4. 10.7% of power is contained at discrete frequencies, including 10% at d-c.

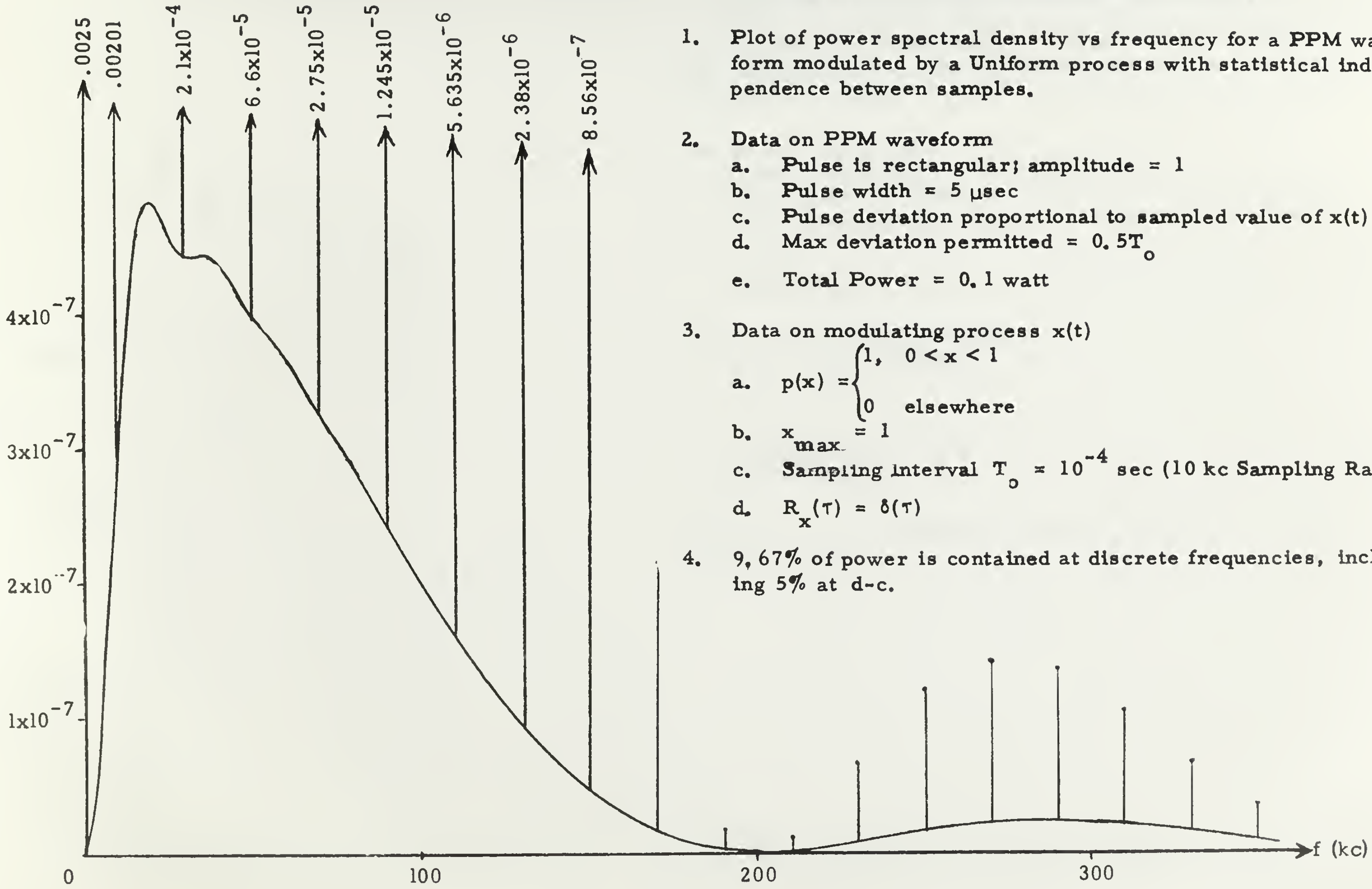


Figure 7-7. Spectrum for Uniform Case #3.

1. Plot of power spectral density vs frequency for a PPM waveform modulated by a Uniform process with statistical independence between samples.

2. Data on PPM waveform
   a. Pulse is rectangular; amplitude = 1
   b. Pulse width = 5 μsec
   c. Pulse deviation proportional to sampled value of x(t)
   d. Max deviation permitted = $0.5T_o$
   e. Total Power = 0.1 watt

3. Data on modulating process x(t)
   a. $p(x) = \begin{cases} 1, & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$
   b. $x_{max} = 1$
   c. Sampling interval $T_o = 10^{-4}$ sec (10 kc Sampling Rate)
   d. $R_x(\tau) = \delta(\tau)$

4. 9.67% of power is contained at discrete frequencies, including 5% at d-c.

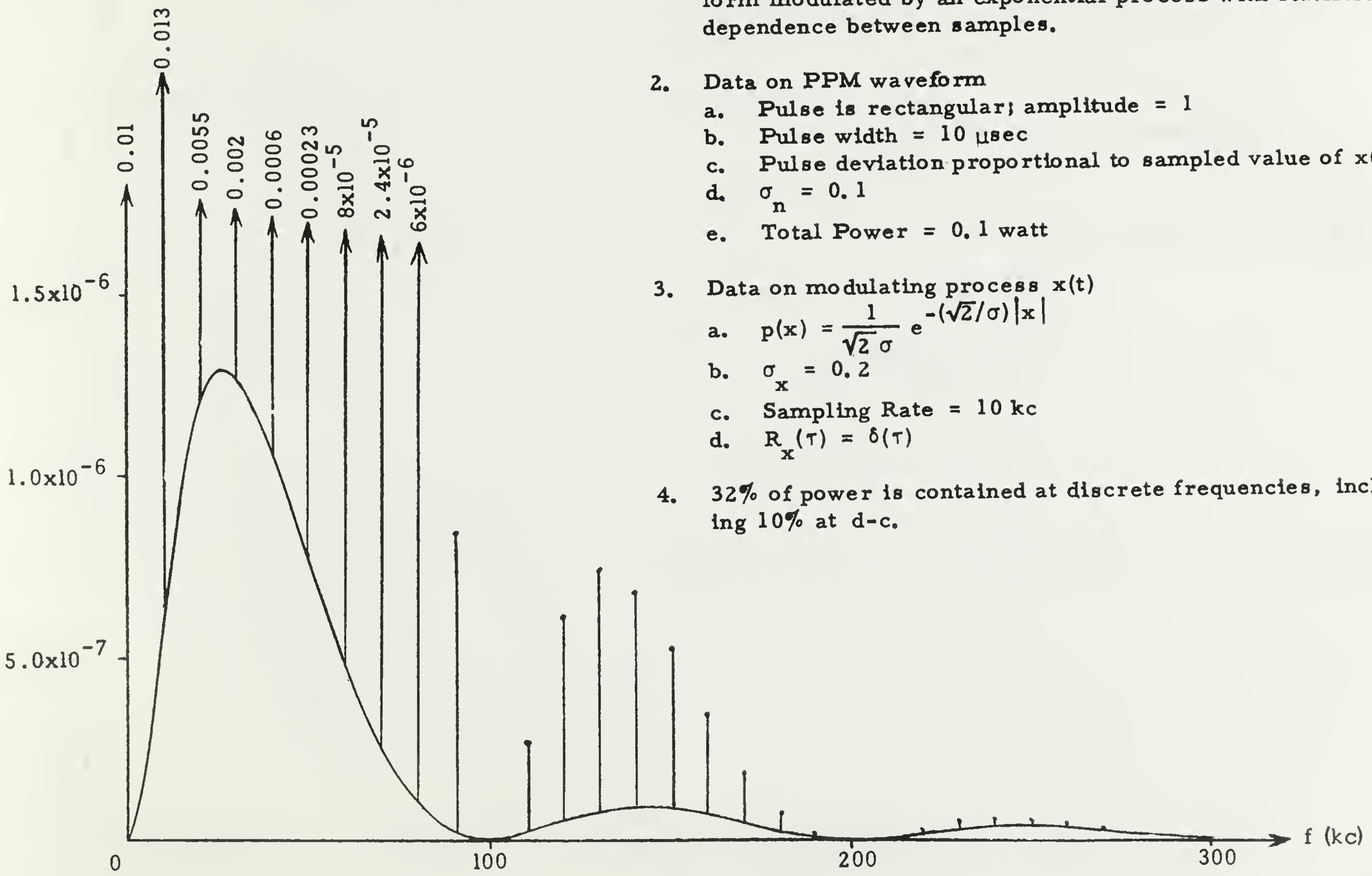


Figure 7-8. Spectrum for Exponential Case #1.

1. Plot of power spectral density vs frequency for a PPM waveform modulated by an exponential process with statistical independence between samples.

2. Data on PPM waveform
   a. Pulse is rectangular; amplitude = 1
   b. Pulse width = 10 μsec
   c. Pulse deviation proportional to sampled value of x(t)
   d. $\sigma_n = 0.1$
   e. Total Power = 0.1 watt

3. Data on modulating process x(t)
   a. $p(x) = \frac{1}{\sqrt{2}\,\sigma} e^{-(\sqrt{2}/\sigma)|x|}$
   b. $\sigma_x = 0.2$
   c. Sampling Rate = 10 kc
   d. $R_x(\tau) = \delta(\tau)$

4. 32% of power is contained at discrete frequencies, including 10% at d-c.

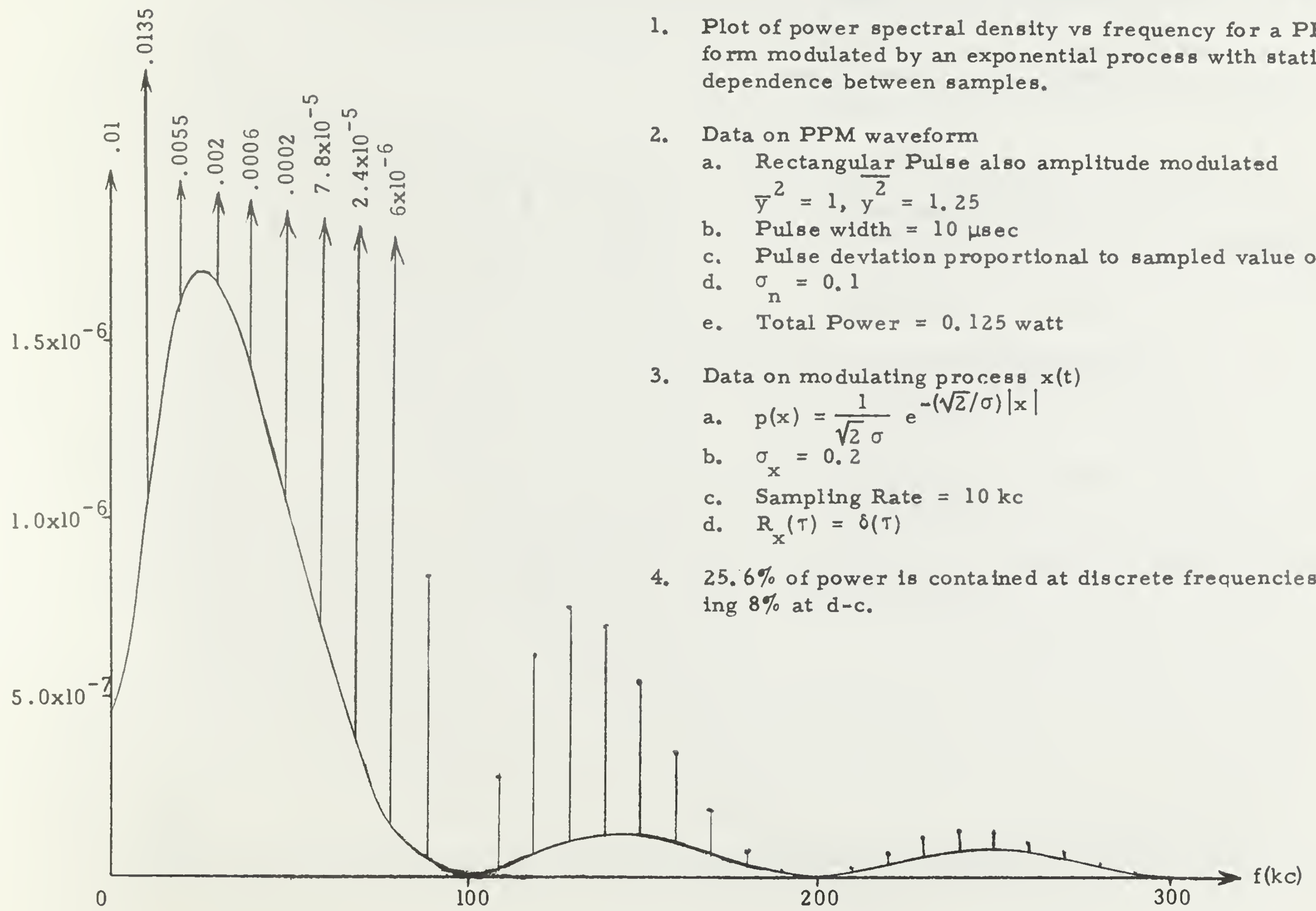


Figure 7-9. Spectrum for Exponential Case #2.

1. Plot of power spectral density vs frequency for a PPM waveform modulated by an exponential process with statistical independence between samples.

2. Data on PPM waveform
   a. Rectangular Pulse also amplitude modulated
      $\overline{y}^2 = 1$, $\overline{y^2} = 1.25$
   b. Pulse width = 10 μsec
   c. Pulse deviation proportional to sampled value of x(t)
   d. $\sigma_n = 0.1$
   e. Total Power = 0.125 watt

3. Data on modulating process x(t)
   a. $p(x) = \frac{1}{\sqrt{2}\,\sigma} e^{-(\sqrt{2}/\sigma)|x|}$
   b. $\sigma_x = 0.2$
   c. Sampling Rate = 10 kc
   d. $R_x(\tau) = \delta(\tau)$

4. 25.6% of power is contained at discrete frequencies, including 8% at d-c.

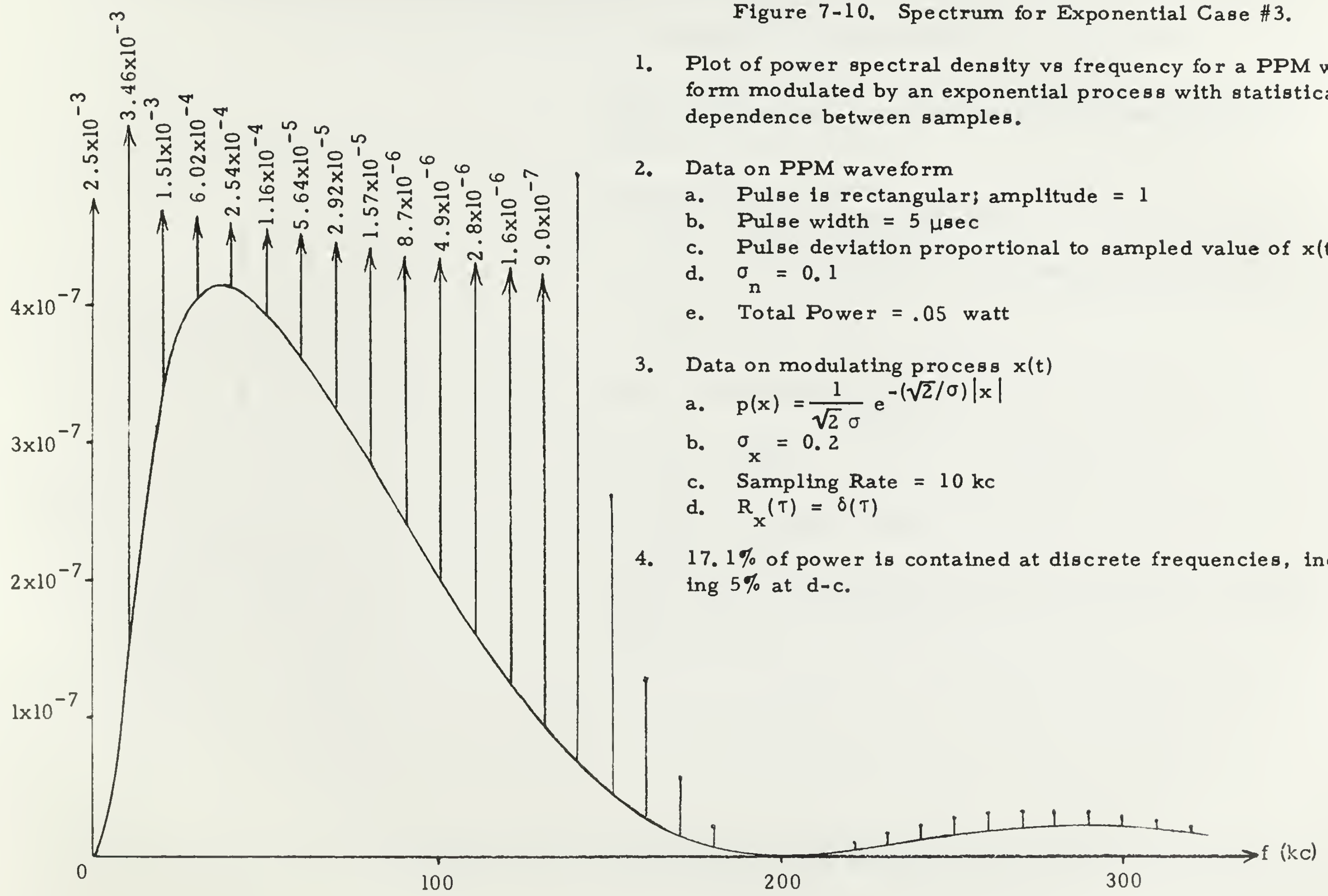


Figure 7-10. Spectrum for Exponential Case #3.

1. Plot of power spectral density vs frequency for a PPM waveform modulated by an exponential process with statistical independence between samples.

2. Data on PPM waveform
   a. Pulse is rectangular; amplitude = 1
   b. Pulse width = 5 μsec
   c. Pulse deviation proportional to sampled value of x(t)
   d. $\sigma_n = 0.1$
   e. Total Power = .05 watt

3. Data on modulating process x(t)
   a. $p(x) = \frac{1}{\sqrt{2}\,\sigma} e^{-(\sqrt{2}/\sigma)|x|}$
   b. $\sigma_x = 0.2$
   c. Sampling Rate = 10 kc
   d. $R_x(\tau) = \delta(\tau)$

4. 17.1% of power is contained at discrete frequencies, including 5% at d-c.

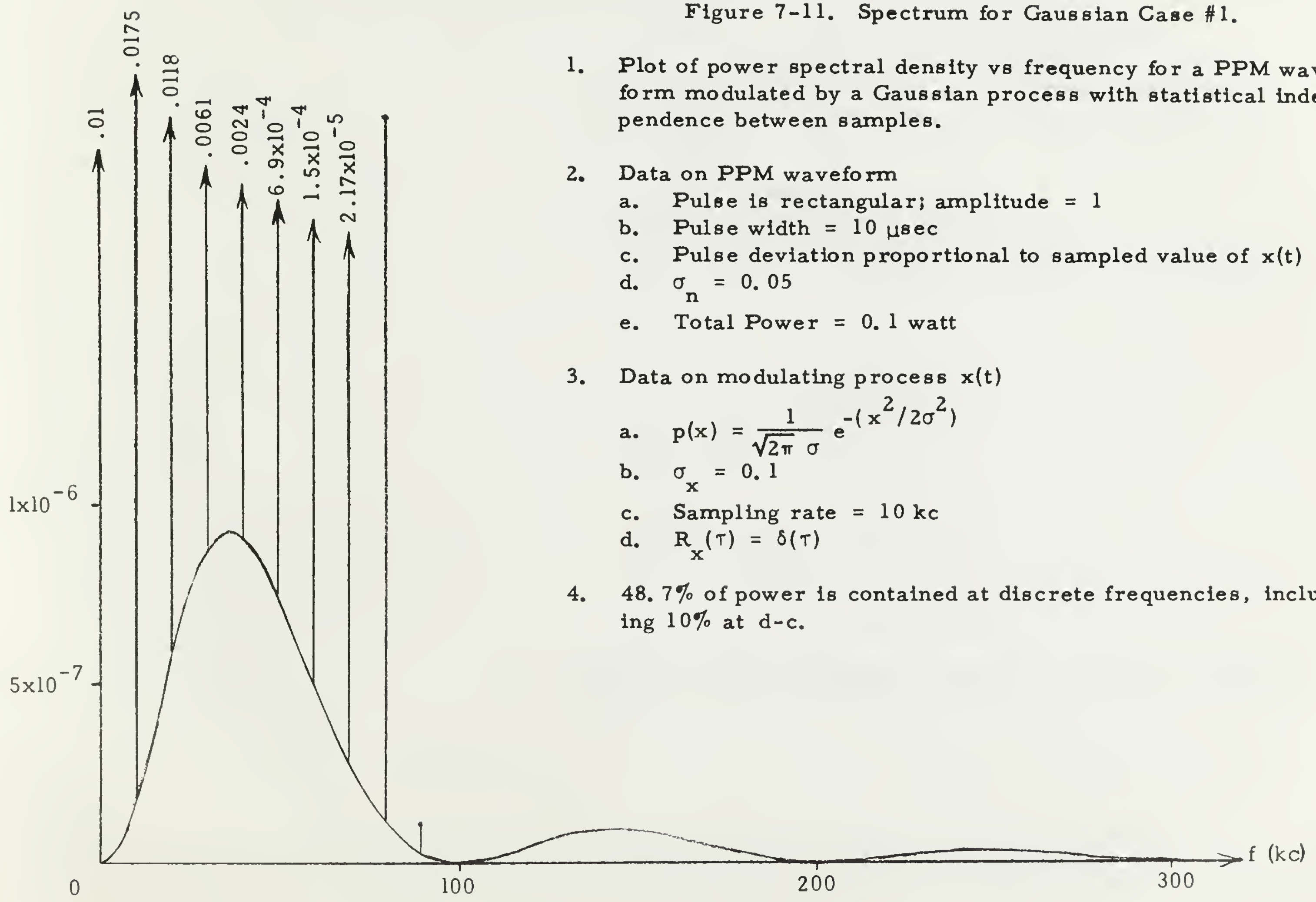


Figure 7-11. Spectrum for Gaussian Case #1.

1. Plot of power spectral density vs frequency for a PPM waveform modulated by a Gaussian process with statistical independence between samples.

2. Data on PPM waveform
   a. Pulse is rectangular; amplitude = 1
   b. Pulse width = 10 μsec
   c. Pulse deviation proportional to sampled value of x(t)
   d. $\sigma_n = 0.05$
   e. Total Power = 0.1 watt

3. Data on modulating process x(t)
   a. $p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x^2/2\sigma^2)}$
   b. $\sigma_x = 0.1$
   c. Sampling rate = 10 kc
   d. $R_x(\tau) = \delta(\tau)$

4. 48.7% of power is contained at discrete frequencies, including 10% at d-c.

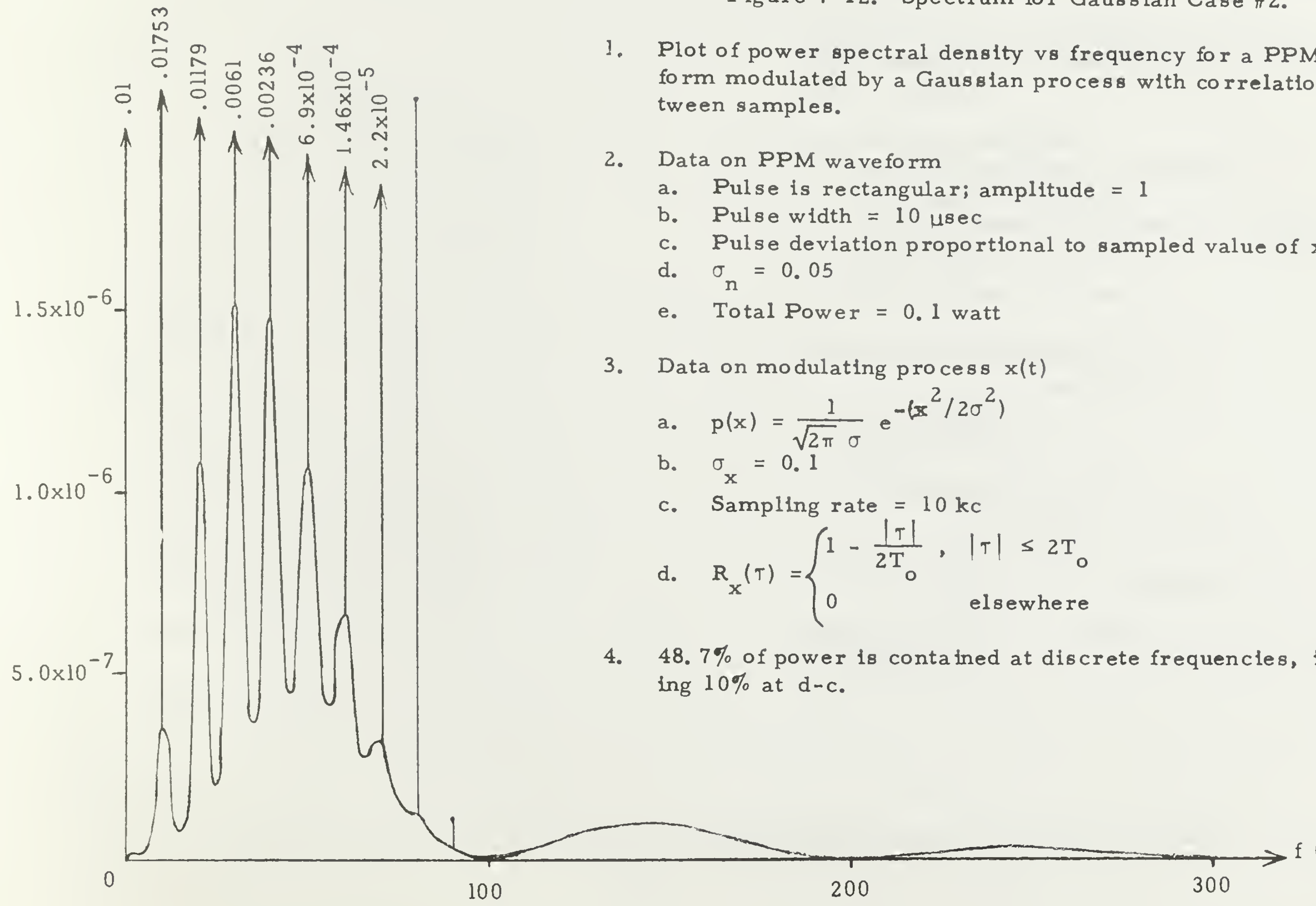


Figure 7-12. Spectrum for Gaussian Case #2.

1. Plot of power spectral density vs frequency for a PPM waveform modulated by a Gaussian process with correlation between samples.

2. Data on PPM waveform
   a. Pulse is rectangular; amplitude = 1
   b. Pulse width = 10 μsec
   c. Pulse deviation proportional to sampled value of x(t)
   d. $\sigma_n = 0.05$
   e. Total Power = 0.1 watt

3. Data on modulating process x(t)
   a. $p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x^2/2\sigma^2)}$
   b. $\sigma_x = 0.1$
   c. Sampling rate = 10 kc
   d. $R_x(\tau) = \begin{cases} 1 - \frac{|\tau|}{2T_o}, & |\tau| \le 2T_o \\ 0 & \text{elsewhere} \end{cases}$

4. 48.7% of power is contained at discrete frequencies, including 10% at d-c.

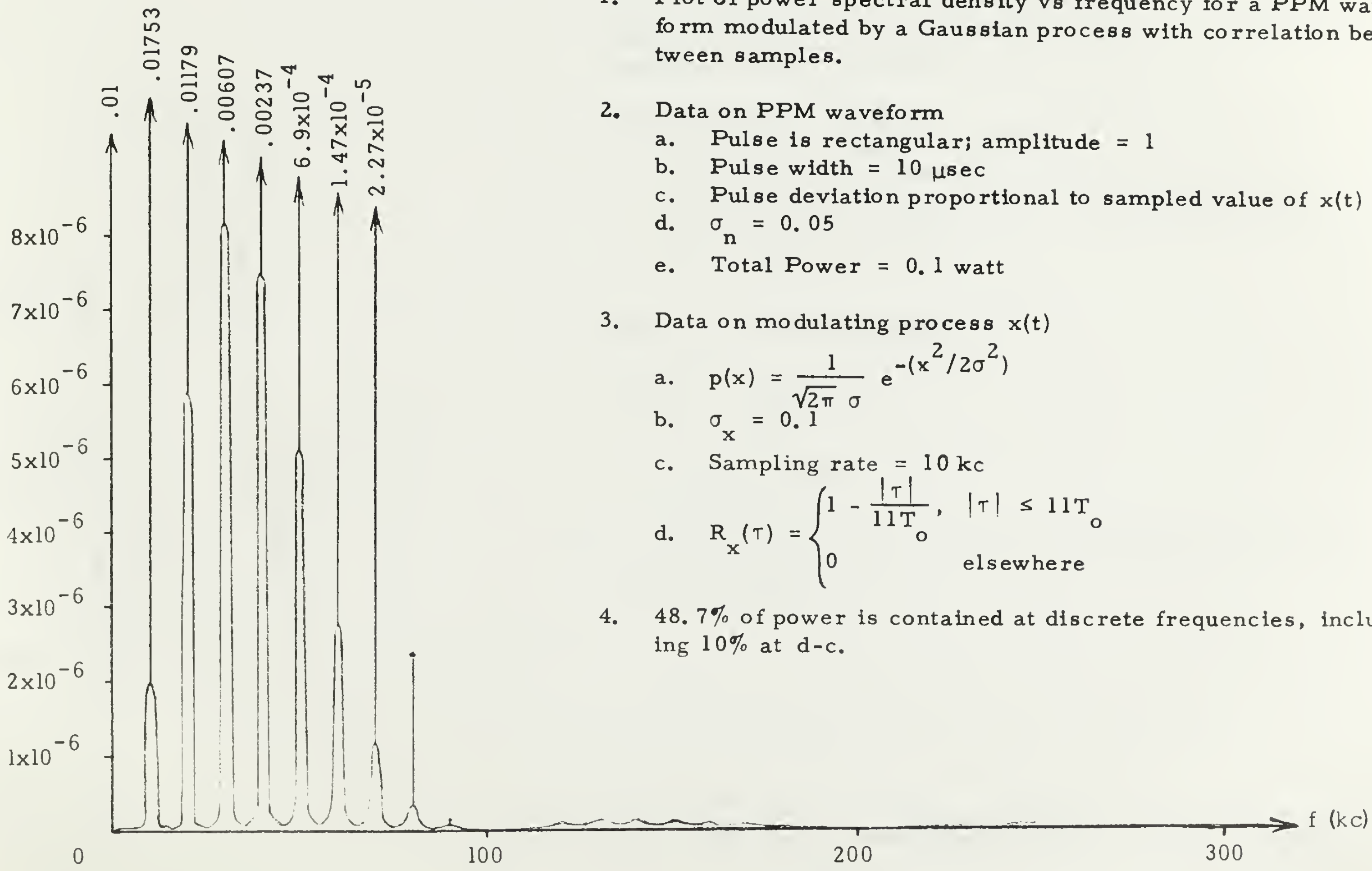


Figure 7-13. Spectrum for Gaussian Case #3.

1. Plot of power spectral density vs frequency for a PPM waveform modulated by a Gaussian process with correlation between samples.

2. Data on PPM waveform
   a. Pulse is rectangular; amplitude = 1
   b. Pulse width = 10 μsec
   c. Pulse deviation proportional to sampled value of $x(t)$
   d. $\sigma_n = 0.05$
   e. Total Power = 0.1 watt

3. Data on modulating process $x(t)$
   a. $p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x^2/2\sigma^2)}$
   b. $\sigma_x = 0.1$
   c. Sampling rate = 10 kc
   d. $R_x(\tau) = \begin{cases} 1 - \frac{|\tau|}{11T_o}, & |\tau| \le 11T_o \\ 0 & \text{elsewhere} \end{cases}$

4. 48.7% of power is contained at discrete frequencies, including 10% at d-c.

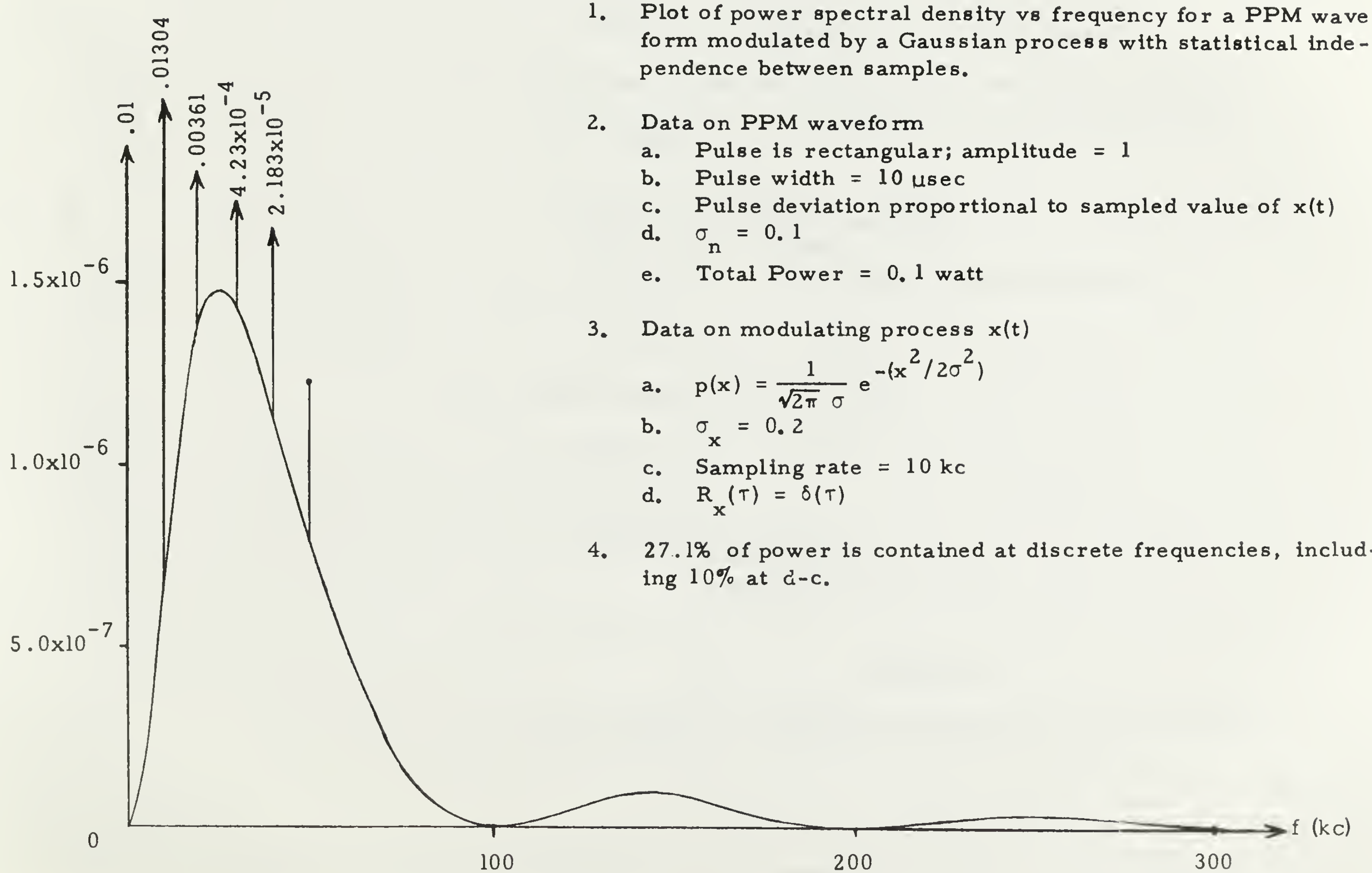


Figure 7-14. Spectrum for Gaussian Case #4.

1. Plot of power spectral density vs frequency for a PPM waveform modulated by a Gaussian process with statistical independence between samples.

2. Data on PPM waveform
   a. Pulse is rectangular; amplitude = 1
   b. Pulse width = 10 μsec
   c. Pulse deviation proportional to sampled value of x(t)
   d. $\sigma_n = 0.1$
   e. Total Power = 0.1 watt

3. Data on modulating process x(t)
   a. $p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x^2/2\sigma^2)}$
   b. $\sigma_x = 0.2$
   c. Sampling rate = 10 kc
   d. $R_x(\tau) = \delta(\tau)$

4. 27.1% of power is contained at discrete frequencies, including 10% at d-c.

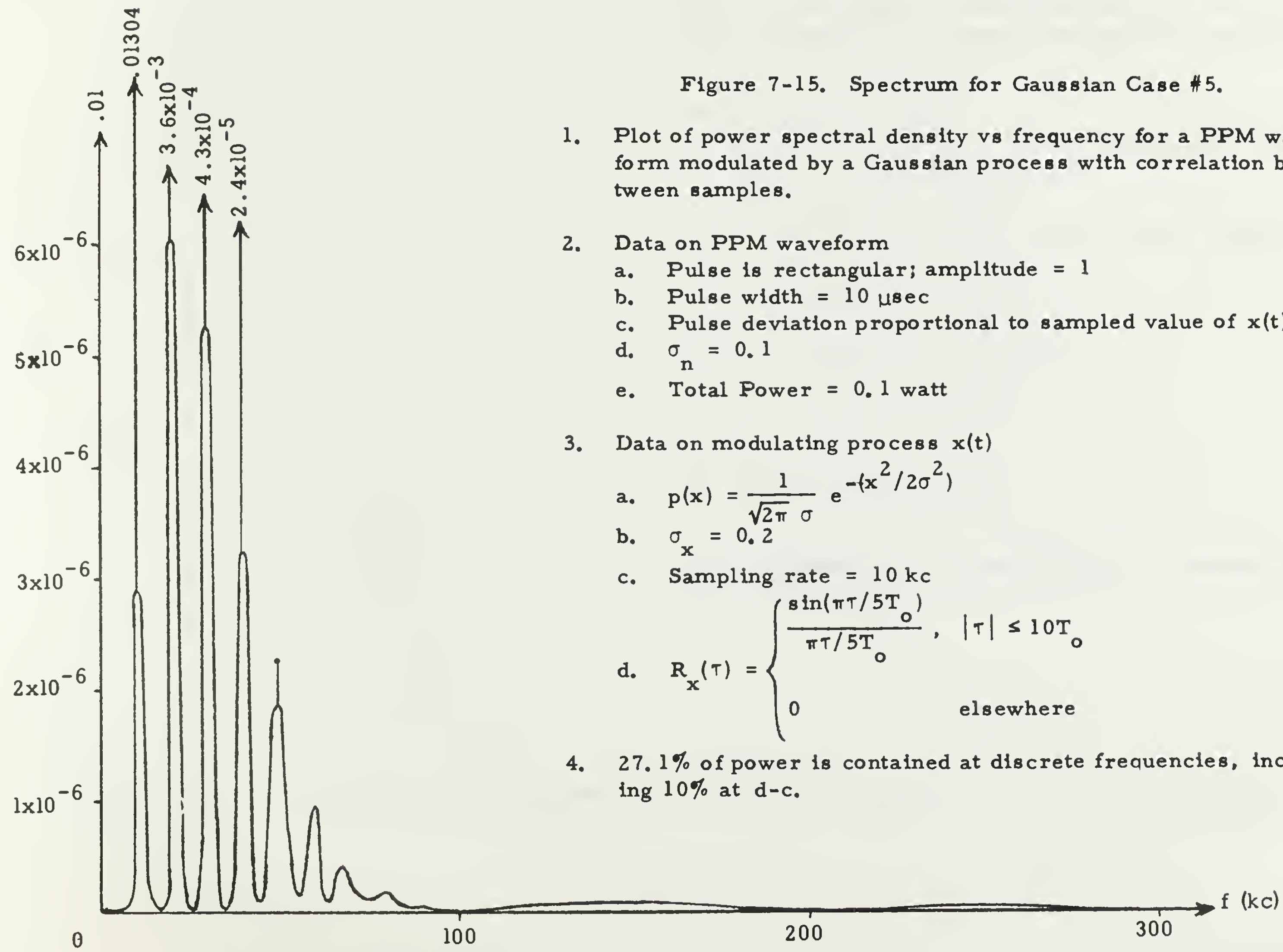


Figure 7-15. Spectrum for Gaussian Case #5.

1. Plot of power spectral density vs frequency for a PPM waveform modulated by a Gaussian process with correlation between samples.

2. Data on PPM waveform
   a. Pulse is rectangular; amplitude = 1
   b. Pulse width = 10 μsec
   c. Pulse deviation proportional to sampled value of x(t)
   d. $\sigma_n = 0.1$
   e. Total Power = 0.1 watt

3. Data on modulating process x(t)
   a. $p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x^2/2\sigma^2)}$
   b. $\sigma_x = 0.2$
   c. Sampling rate = 10 kc
   d. $R_x(\tau) = \begin{cases} \frac{\sin(\pi\tau/5T_o)}{\pi\tau/5T_o}, & |\tau| \le 10T_o \\ 0 & \text{elsewhere} \end{cases}$

4. 27.1% of power is contained at discrete frequencies, including 10% at d-c.

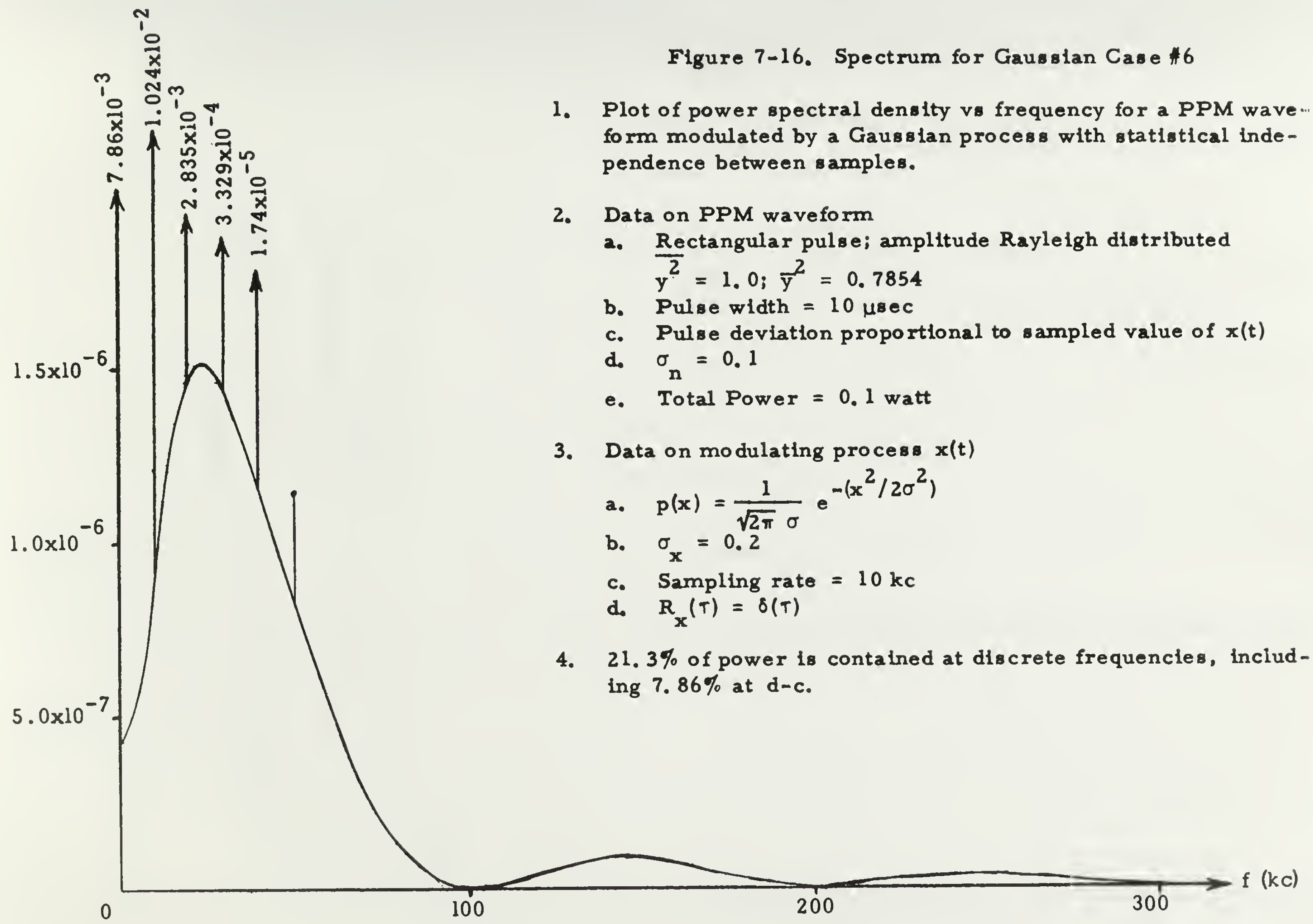


Figure 7-16. Spectrum for Gaussian Case #6

1. Plot of power spectral density vs frequency for a PPM waveform modulated by a Gaussian process with statistical independence between samples.

2. Data on PPM waveform
   a. Rectangular pulse; amplitude Rayleigh distributed
      $\overline{y^2} = 1.0$; $\overline{y}^2 = 0.7854$
   b. Pulse width = 10 μsec
   c. Pulse deviation proportional to sampled value of x(t)
   d. $\sigma_n = 0.1$
   e. Total Power = 0.1 watt

3. Data on modulating process x(t)
   a. $p(x) = \dfrac{1}{\sqrt{2\pi}\,\sigma} e^{-(x^2/2\sigma^2)}$
   b. $\sigma_x = 0.2$
   c. Sampling rate = 10 kc
   d. $R_x(\tau) = \delta(\tau)$

4. 21.3% of power is contained at discrete frequencies, including 7.86% at d-c.

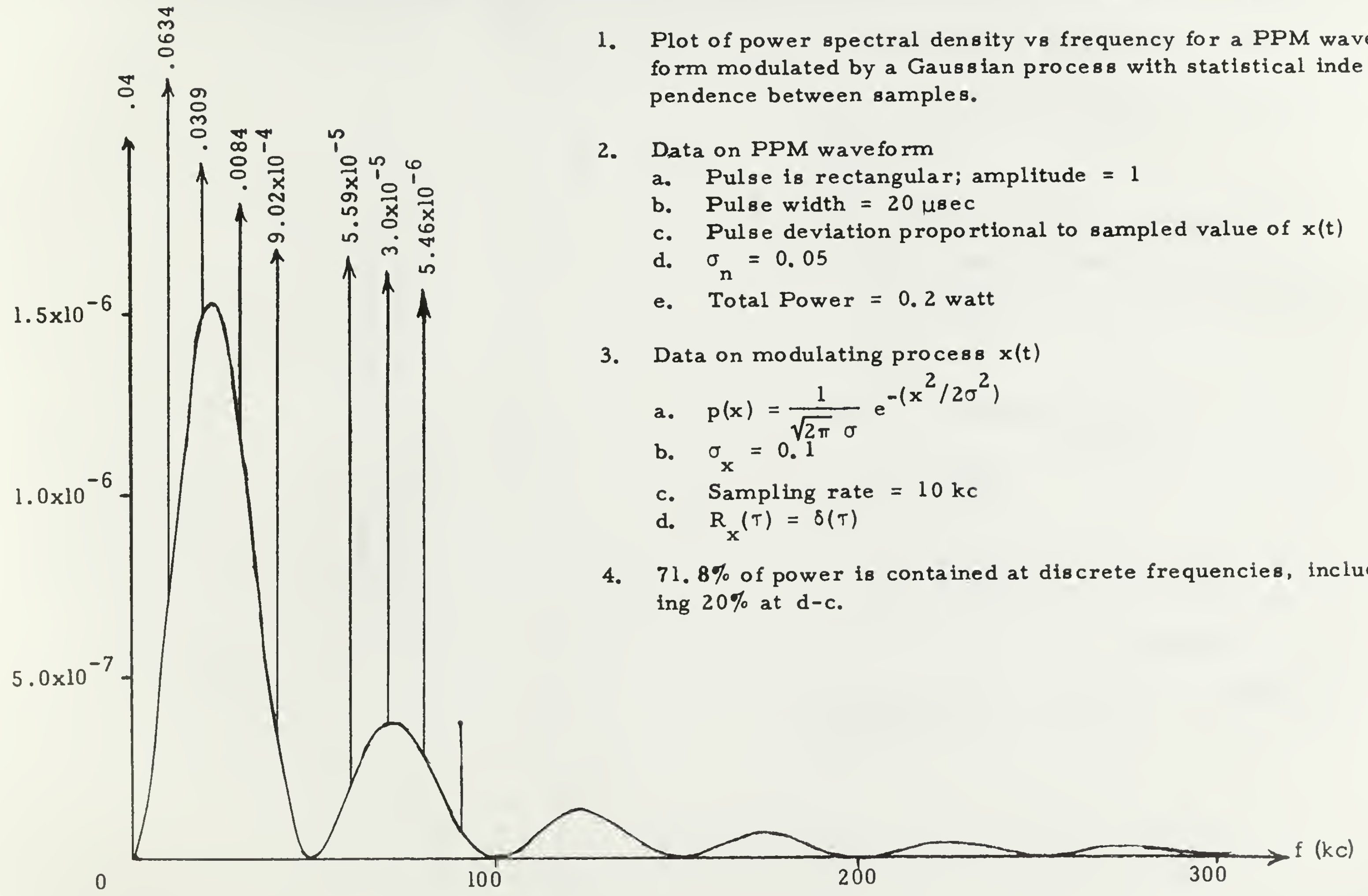
Figure 7-17. Spectrum for Gaussian Case #7.

1. Plot of power spectral density vs frequency for a PPM waveform modulated by a Gaussian process with statistical independence between samples.

2. Data on PPM waveform
   a. Pulse is rectangular; amplitude = 1
   b. Pulse width = 20 μsec
   c. Pulse deviation proportional to sampled value of x(t)
   d. $\sigma_n = 0.05$
   e. Total Power = 0.2 watt

3. Data on modulating process x(t)
   a. $p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x^2/2\sigma^2)}$
   b. $\sigma_x = 0.1$
   c. Sampling rate = 10 kc
   d. $R_x(\tau) = \delta(\tau)$

4. 71.8% of power is contained at discrete frequencies, including 20% at d-c.



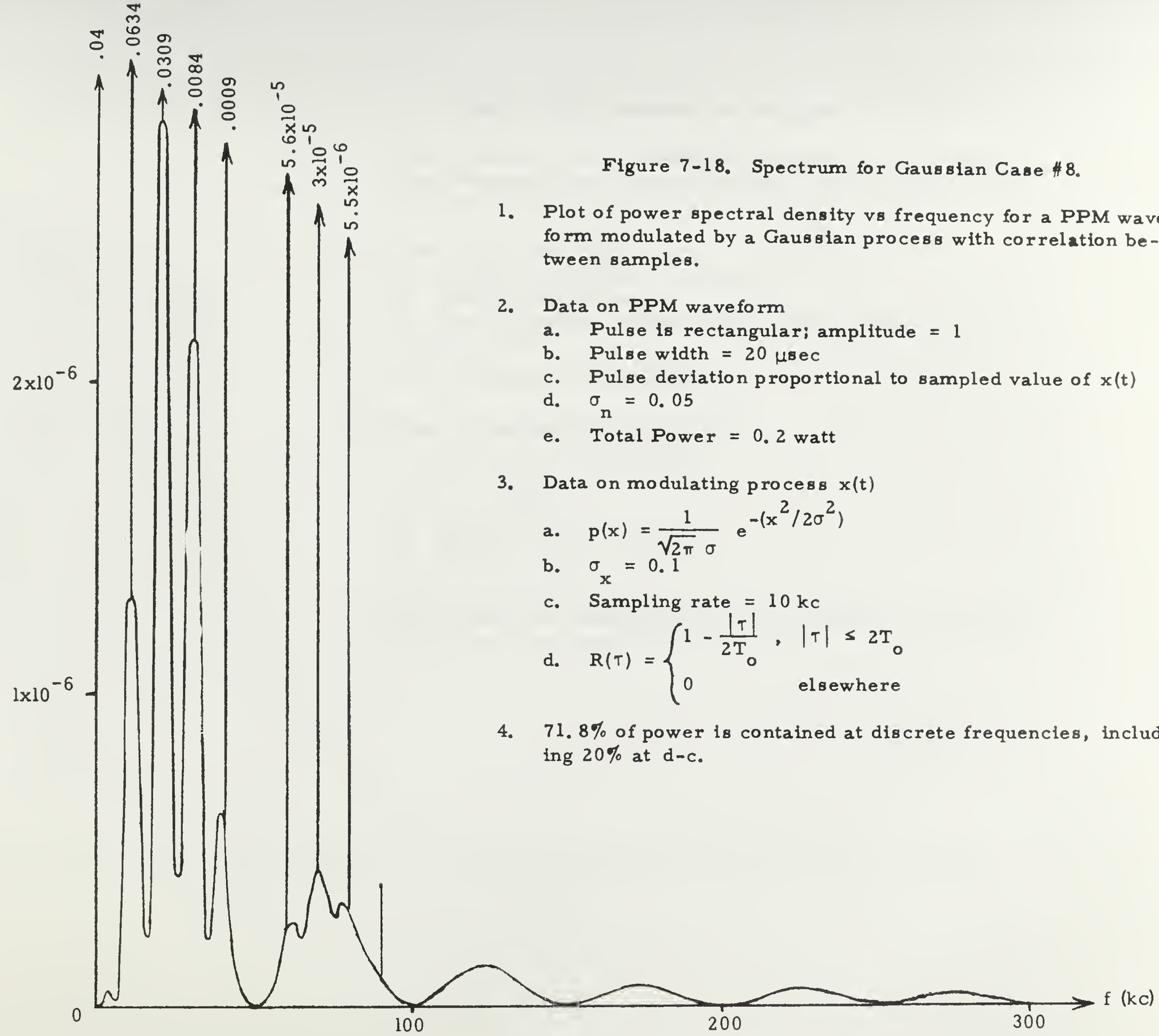


Figure 7-18. Spectrum for Gaussian Case #8.

1. Plot of power spectral density vs frequency for a PPM waveform modulated by a Gaussian process with correlation between samples.

2. Data on PPM waveform
   a. Pulse is rectangular; amplitude = 1
   b. Pulse width = 20 μsec
   c. Pulse deviation proportional to sampled value of x(t)
   d. $\sigma_n = 0.05$
   e. Total Power = 0.2 watt

3. Data on modulating process x(t)
   a. $p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x^2/2\sigma^2)}$
   b. $\sigma_x = 0.1$
   c. Sampling rate = 10 kc
   d. $R(\tau) = \begin{cases} 1 - \frac{|\tau|}{2T_o}, & |\tau| \leq 2T_o \\ 0 & \text{elsewhere} \end{cases}$

4. 71.8% of power is contained at discrete frequencies, including 20% at d-c.

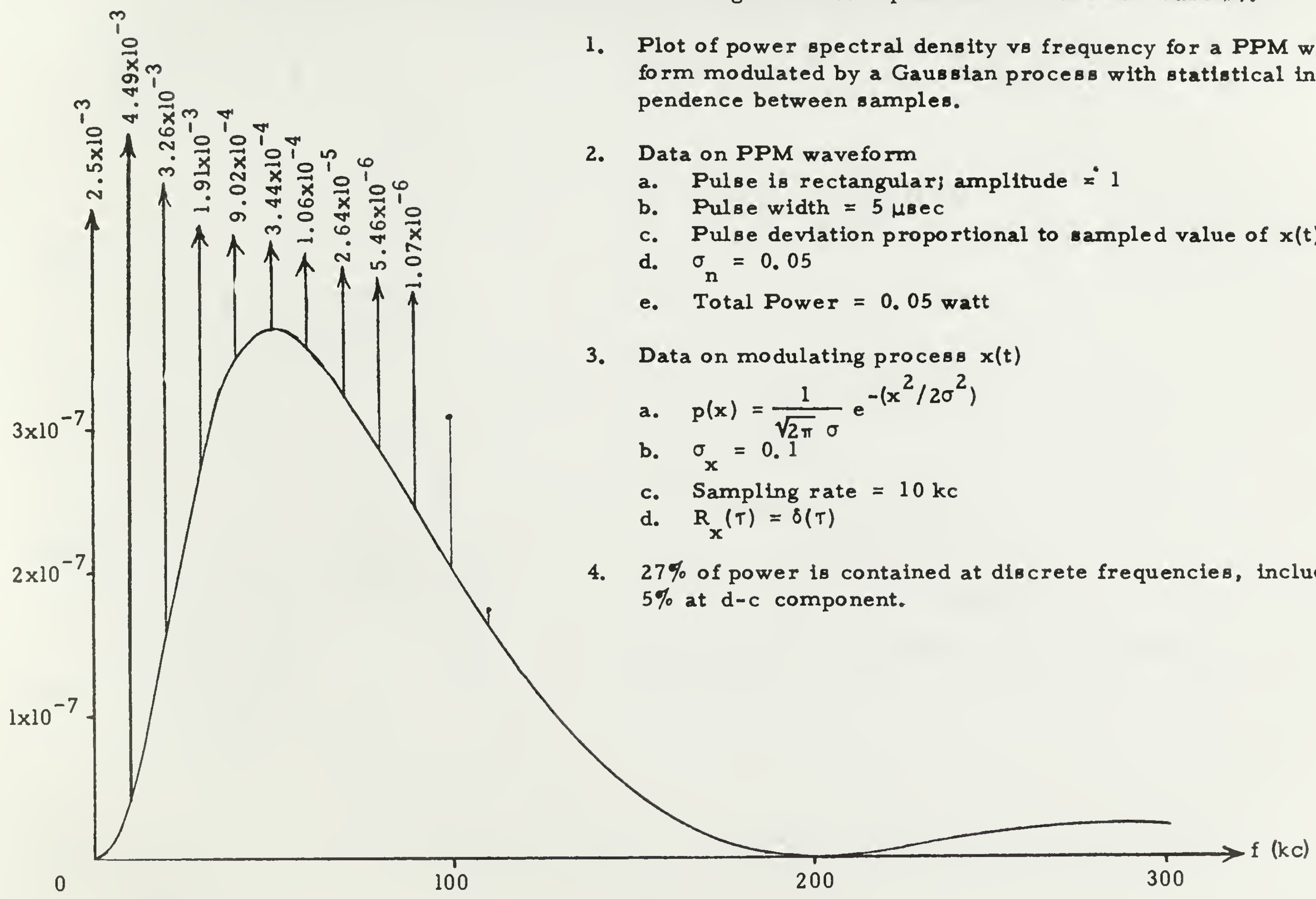


Figure 7-19. Spectrum for Gaussian Case #9.

1. Plot of power spectral density vs frequency for a PPM waveform modulated by a Gaussian process with statistical independence between samples.

2. Data on PPM waveform
   a. Pulse is rectangular; amplitude $\doteq 1$
   b. Pulse width = 5 µsec
   c. Pulse deviation proportional to sampled value of x(t)
   d. $\sigma_n = 0.05$
   e. Total Power = 0.05 watt

3. Data on modulating process x(t)
   a. $p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x^2/2\sigma^2)}$
   b. $\sigma_x = 0.1$
   c. Sampling rate = 10 kc
   d. $R_x(\tau) = \delta(\tau)$

4. 27% of power is contained at discrete frequencies, including 5% at d-c component.

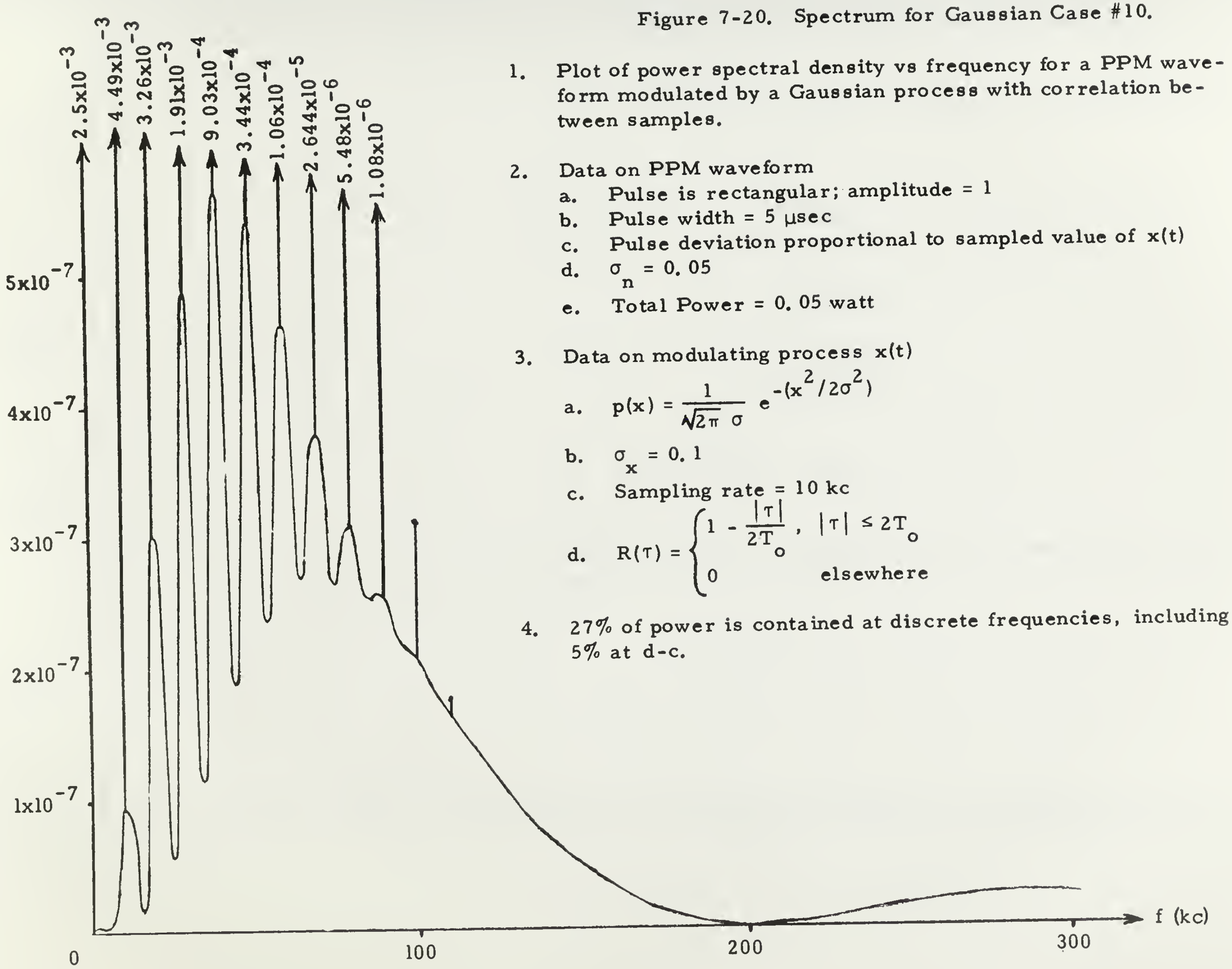


Figure 7-20. Spectrum for Gaussian Case #10.

1. Plot of power spectral density vs frequency for a PPM waveform modulated by a Gaussian process with correlation between samples.

2. Data on PPM waveform
   a. Pulse is rectangular; amplitude = 1
   b. Pulse width = 5 μsec
   c. Pulse deviation proportional to sampled value of $x(t)$
   d. $\sigma_n = 0.05$
   e. Total Power = 0.05 watt

3. Data on modulating process $x(t)$
   a. $p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x^2/2\sigma^2)}$
   b. $\sigma_x = 0.1$
   c. Sampling rate = 10 kc
   d. $R(\tau) = \begin{cases} 1 - \frac{|\tau|}{2T_o}, & |\tau| \le 2T_o \\ 0 & \text{elsewhere} \end{cases}$

4. 27% of power is contained at discrete frequencies, including 5% at d-c.

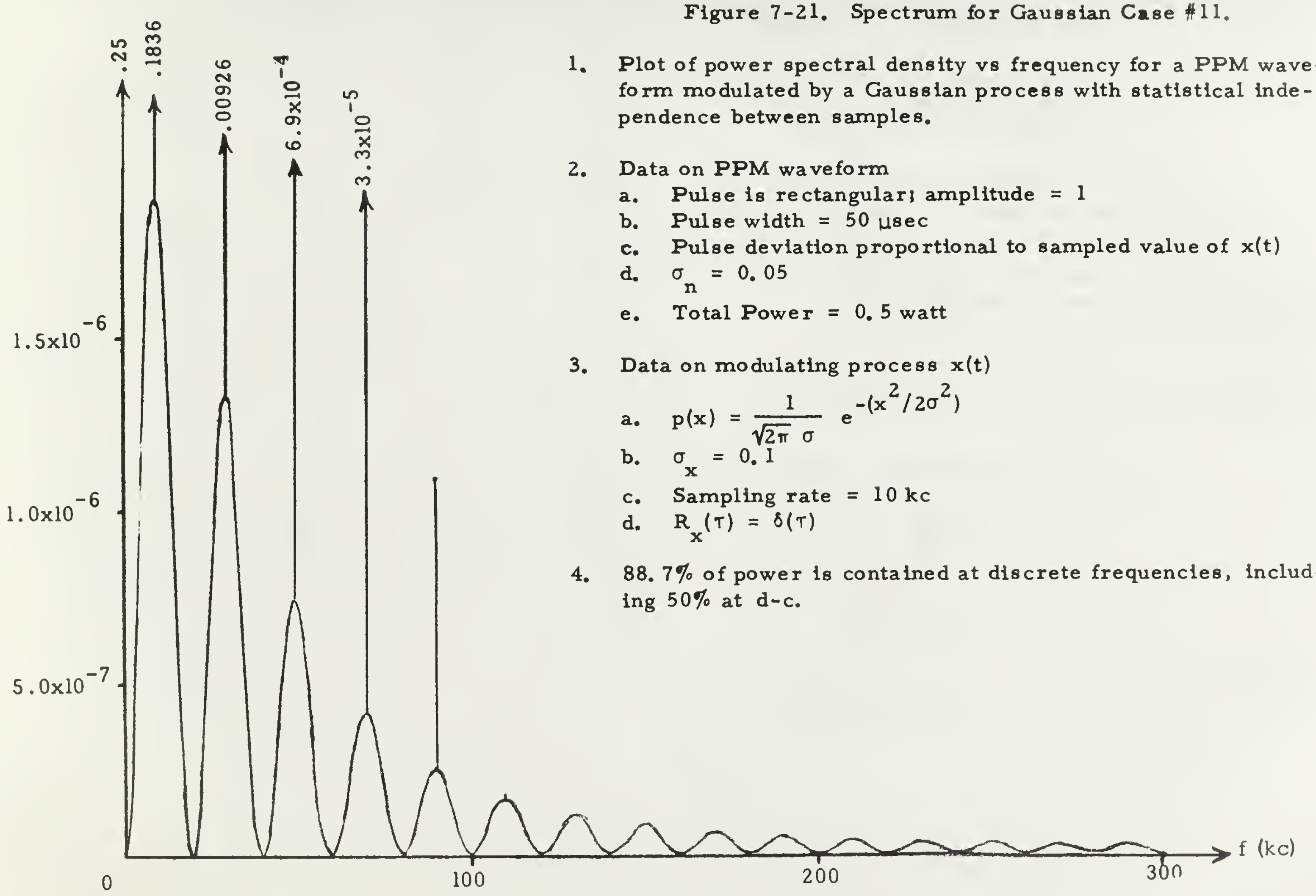


Figure 7-21. Spectrum for Gaussian Case #11.

1. Plot of power spectral density vs frequency for a PPM waveform modulated by a Gaussian process with statistical independence between samples.

2. Data on PPM waveform
   a. Pulse is rectangular; amplitude = 1
   b. Pulse width = 50 μsec
   c. Pulse deviation proportional to sampled value of $x(t)$
   d. $\sigma_n = 0.05$
   e. Total Power = 0.5 watt

3. Data on modulating process $x(t)$
   a. $p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x^2/2\sigma^2)}$
   b. $\sigma_x = 0.1$
   c. Sampling rate = 10 kc
   d. $R_x(\tau) = \delta(\tau)$

4. 88.7% of power is contained at discrete frequencies, including 50% at d-c.

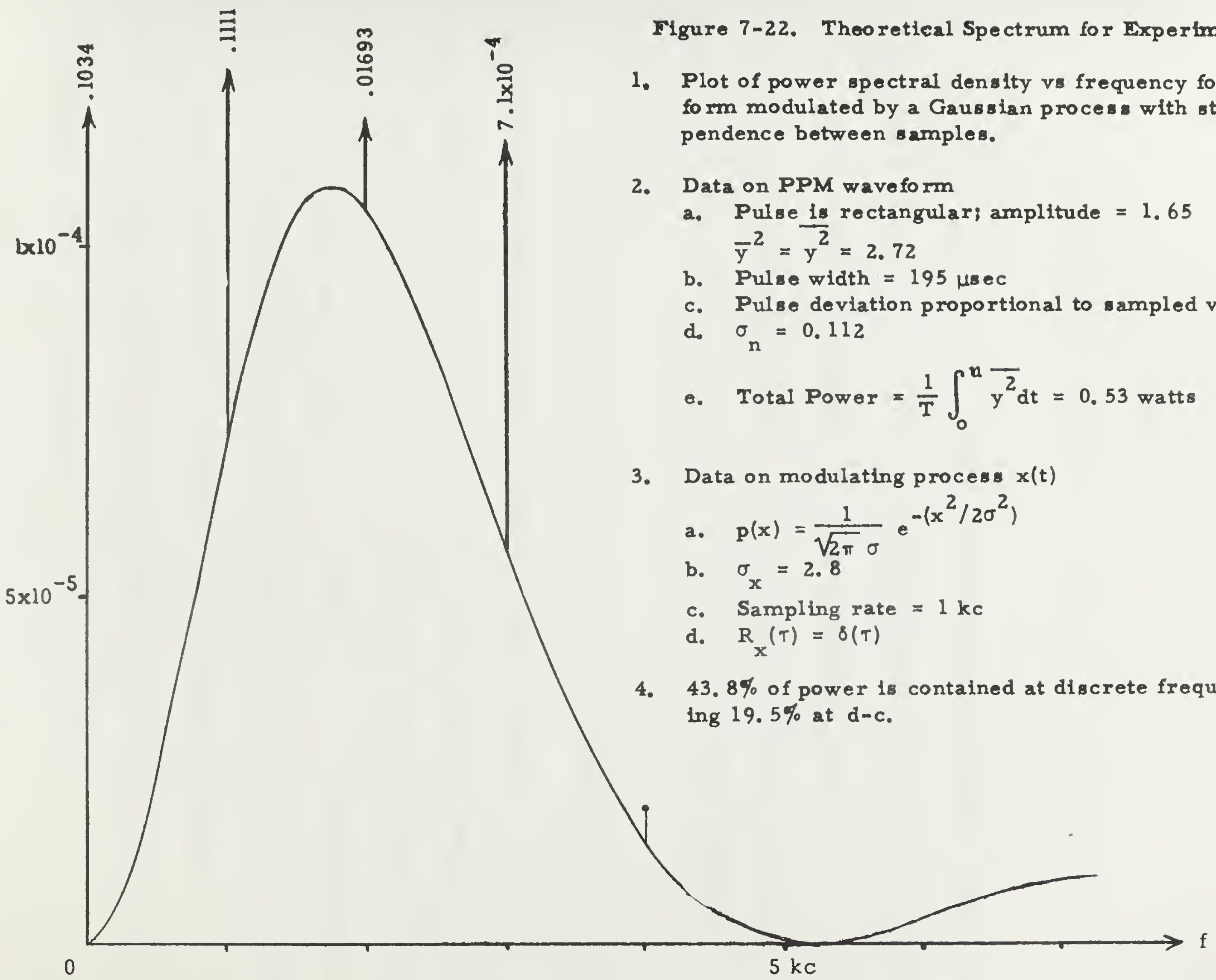


Figure 7-22. Theoretical Spectrum for Experimental Model.

1. Plot of power spectral density vs frequency for a PPM waveform modulated by a Gaussian process with statistical independence between samples.

2. Data on PPM waveform
   a. Pulse is rectangular; amplitude = 1.65
      $\overline{y}^2 = \overline{y^2} = 2.72$
   b. Pulse width = 195 μsec
   c. Pulse deviation proportional to sampled value of x(t)
   d. $\sigma_n = 0.112$

   e. Total Power $= \frac{1}{T}\int_0^u \overline{y^2}\,dt = 0.53$ watts

3. Data on modulating process x(t)
   a. $p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x^2/2\sigma^2)}$
   b. $\sigma_x = 2.8$
   c. Sampling rate = 1 kc
   d. $R_x(\tau) = \delta(\tau)$

4. 43.8% of power is contained at discrete frequencies, including 19.5% at d-c.

8. Spectral Analysis of Actual PPM Waveforms

A comparison of the results from the spectral model was made with the results determined experimentally. The objective of this aspect of the investigation was to determine if short time interval analysis would yield a reasonable estimate of the power spectral density of PPM waveforms.

In order to generate PPM waveforms comparable to those associated with the spectral model, a pulse position modulator was designed and built. The circuit and performance characteristics of the modulator are included in Appendix A. The unmodulated pulse train generated has a pulse repetition rate of one kilocycle, and a pulse width of 195 microseconds.

The spectrum analyzer (periodogram calculator) used for the experimental measurements is known by the trade name "Spectran". The Spectran is one of the parallel filter bank types of analyzers described in section 3. It contains 480 parallel filters, and has a resolution of 25 cycles per second (cps). The analysis time is 18 milliseconds. The first two frequency bands of the Spectran were used for the PPM analysis, covering 20 to 3852 and 3852 to 7684 cps, respectively. The use of two bands accounts for what appears to be a slight discontinuity in the pictures to be presented later in this section.

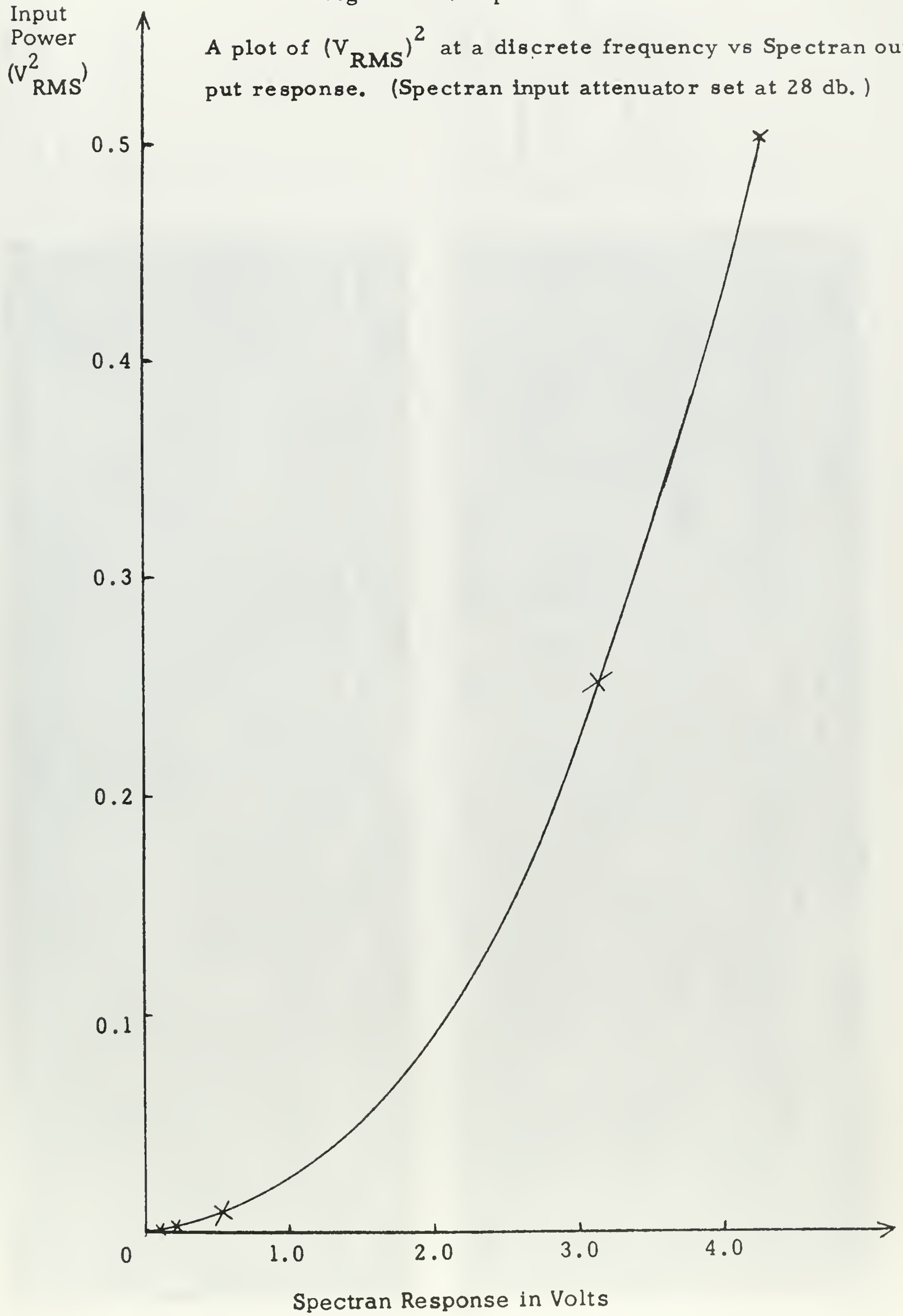
The Spectran was calibrated by inserting a sine wave input of an arbitrary frequency (two kilocycles) and recording the analyzer response for measured power inputs. The input power was considered the square

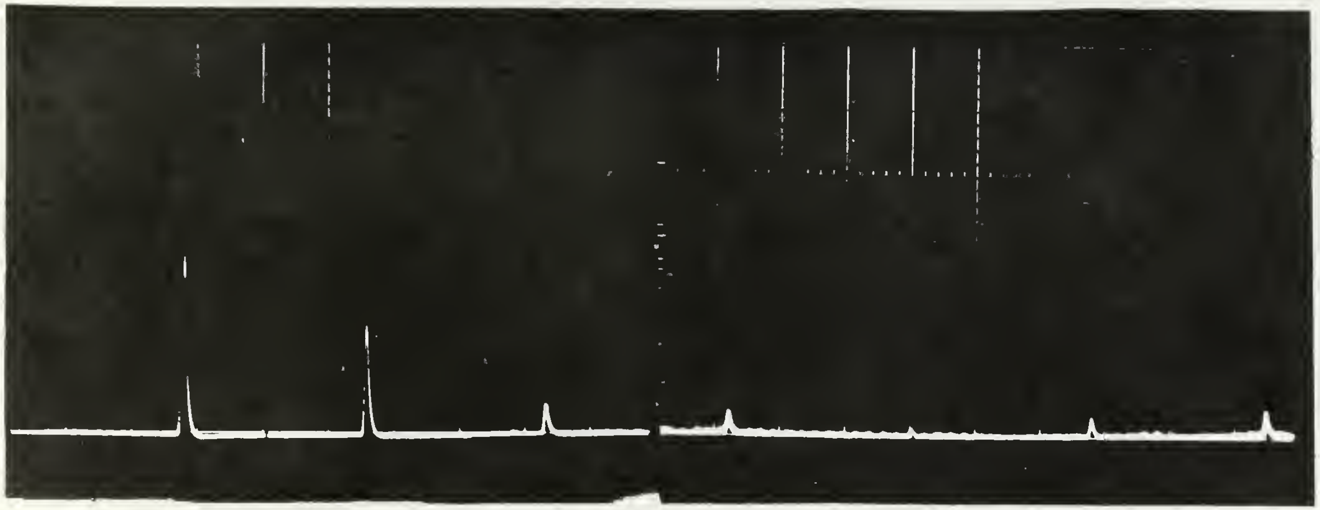
of the RMS voltage input, which is equivalent to the spectral model definition of power into a one ohm resistor. The resulting calibration curve is included as Figure 8-1. Unfortunately, the calibration curve is only accurate for spectra at discrete frequencies (delta functions) as a consequence of having been based upon data taken for discrete frequency inputs. The calibration curve will not yield an accurate value of the continuous spectrum at a specific frequency f_o. Each of the 480 "spectral windows" of the Spectran has a bandwidth of about 10 cps. Therefore, when a continuous spectrum is measured, the measurement obtained at a frequency f_o is actually the proportionate power of <u>all</u> frequencies that are passed through the spectral window that passes the frequency f_o. Further, the noise level of the measuring equipment prohibited accurate measurements of the continuous portion of the spectrum. Consequently, for the experimental results, a determination of the exact values of the continuous spectrum must be attempted by a different procedure. The general shape of the continuous spectrum was determined by the Spectran analysis, however, and is displayed in Figure 8-3. A discussion of the background related to the photographs in Figure 8-3 is considered necessary in order that they may be properly interpreted. As predicted by the computer model, the value of the power spectral density at any frequency of the continuous spectrum is several orders of magnitude less than that of the larger discrete components. In addition, the predicted average levels of the continuous spectrum is about the same as the noise level output of the measuring equipment (the Spectran). Thus, if one

visually subtracts the noise level and the effects of the discrete component as determined from the top picture in Figure 8-3, then a fair estimate of the shape of the continuous spectrum may be gleaned from the bottom picture.

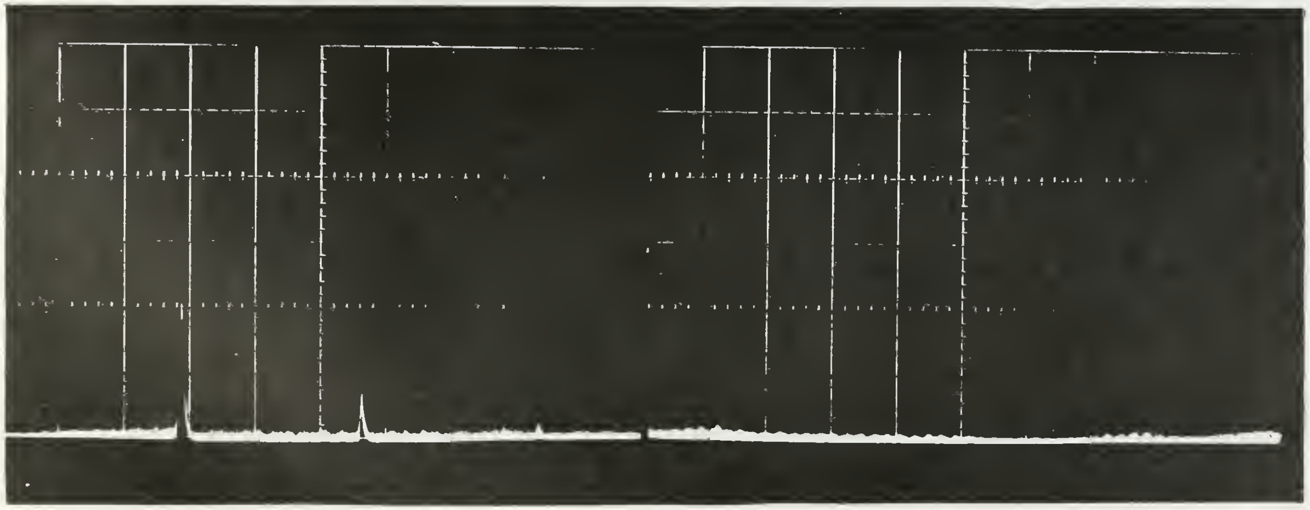
Figure 8-1. Spectran Calibration Curve

A plot of $(V_{RMS})^2$ at a discrete frequency vs Spectran output response. (Spectran input attenuator set at 28 db.)



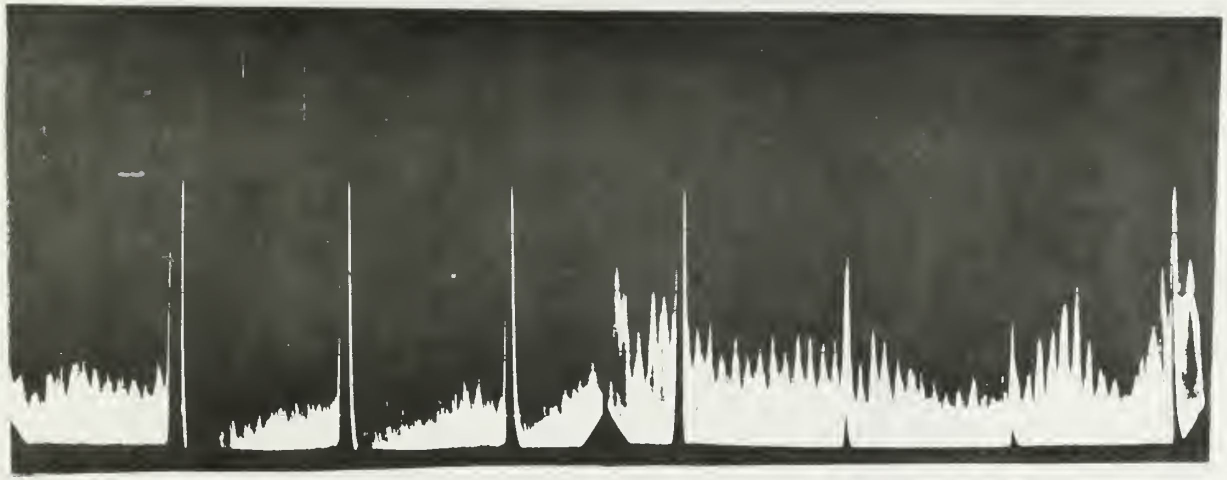


Spectrum of the unmodulated PPM waveform.
55 Sweeps.
Horiz: 361 cps/cm
Vert: 1 V/cm

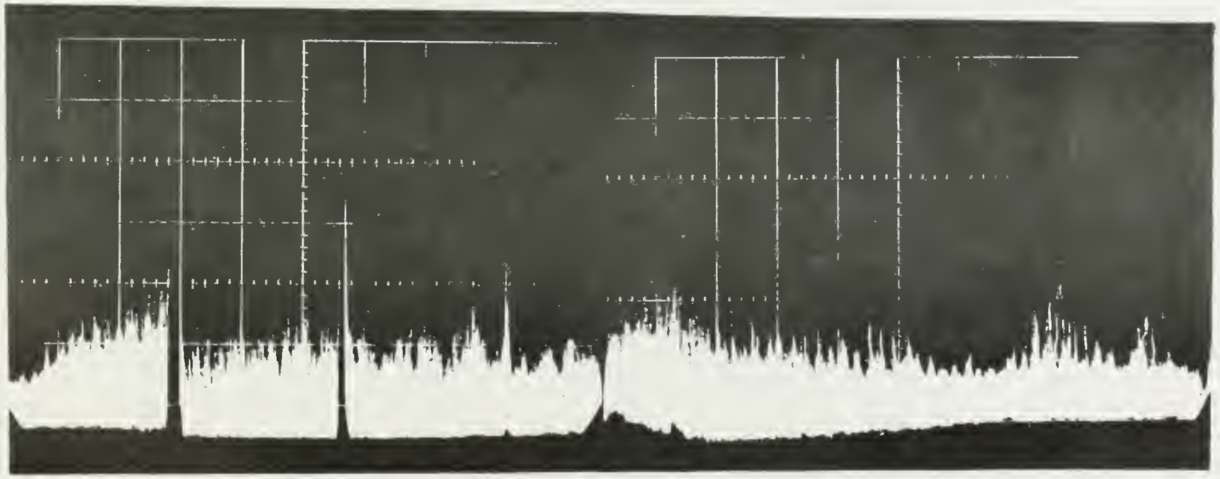


Spectrum of the modulated PPM waveform.
445 Sweeps.
Horiz: 361 cps/cm
Vert: 1 V/cm
Modulation Data:
1. Gaussian Noise
2. $\sigma_n = 0.112$

Figure 8-2. An Experimental Estimate of the Discrete Power Spectral Density.



Spectrum of the
unmodulated PPM
waveform.
334 Sweeps.
Horiz: 361 cps/cm
Vert: 0.05 V/cm



Spectrum of the
modulated PPM
waveform.
390 Sweeps.
Horiz: 361 cps/cm
Vert: 0.05 V/cm
Modulation Data:
1. Gaussian Noise
2. $\sigma_n = 0.112$

Figure 8-3. An Experimental Estimate of the Continuous Power Spectral Density.

In order to measure the value of the continuous power spectrum at discrete frequencies, a wave analyzer with a tunable 6 cycle bandwidth was employed.[1] The PPM waveform, modulated by a random noise (assumed Gaussian), was used as the input to the wave analyzer. True RMS voltage measurements of the analyzer response were made at 100 cycle increments.[2] The block diagram of the experimental setup is shown in Figure 8-4.

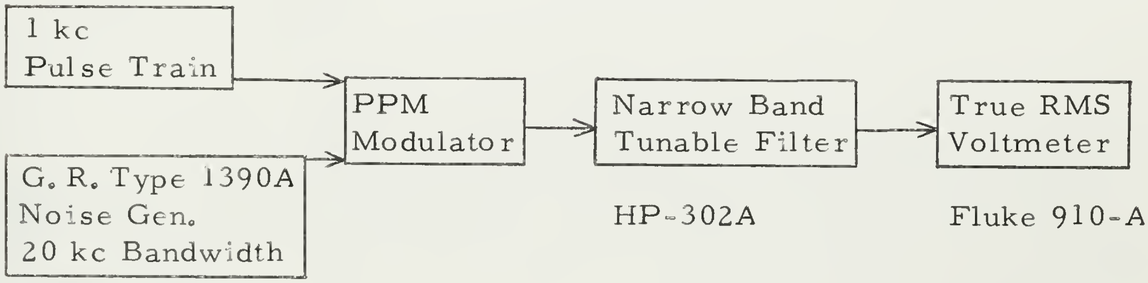


Figure 8-4. Technique for Experimental Measurement of Continuous Spectrum.

Assuming an ideal rectangular response of unit amplitude for the filter, the total power to the RMS voltmeter is

$$\text{Power} = \int_{f_o - 3}^{f_o + 3} S(f)\,df \ ,$$

where f_o is the center frequency of the filter. It shall be assumed that

[1] The analyzer is the HP-302A, which has an equivalent noise bandwidth of 6 cycles with the center frequency adjustable throughout the audio frequency range.

[2] True RMS voltage measurements were made with a "Hi-damped" Fluke Model 910-A voltmeter, which is a power sensitive instrument calibrated in volts RMS.

there are no extreme changes in the power spectral density in any 6 cycle interval; then $S(f)$ may be approximated by a constant $S(f_o)$. The measured power is then approximately

$$\text{Measured Power} = 6S(f_o) .$$

$S(f_o)$ may now be determined from the square of the true RMS voltage measurement, which is the measured power (to a one ohm resistor).

$$S(f_o) = \frac{V_{RMS}^2}{6}$$

A plot of the power spectral density measured in the manner just described is shown in Figure 8-5. The few inconsistent measurements recorded are attributed to this writer's inability to "time average" a slightly fluctuating voltmeter needle by visual observations.[3] For comparison purposes, the theoretical plot of power spectral density is included as the dashed line in Figure 8-5.

The measurement of the discrete spectrum was a simpler matter, as output levels well above the equipment noise levels were being measured. As previously mentioned, the calibration chart applies to the measured results of the discrete spectra, from which may be determined a reasonably accurate measured power level. Unfortunately, the smoothing of the Spectran "readout" is not perfect, resulting in the occurrence

[3] For a discussion of the variance of the output of a square law device (the same type as the measurement fluctuations encountered), see page 256, Davenport, W. B., and W. L. Root, An Introduction to the Theory of Random Signals and Noise, McGraw Hill, 1958.

of peaks and valleys when one sweeps across the spectrum with a constant level input signal. Thus, even the discrete frequency power measurements will have some equipment measuring error introduced.

The discrete spectrum displays are shown in Figure 8-2. From Figure 8-2 and theoretical computations, an experimental versus theoretical comparison was made and is included in Table 8-1. The PPM device is that described in Appendix A modulated with a wide band Gaussian noise source.

Discrete Frequency	Discrete Power Spectral Density			
	Unmodulated Pulse Train		Modulated PPM $\sigma_n = 0.112$; $\bar{y}^2 = \overline{y^2} = 2.72$	
	Experimental (From Fig. 8-2)	Theoretical	Experimental (From Fig. 8-2)	Theoretical
0	No zero frequency response	0.1034	No zero frequency response	0.1034
1 kc	0.195	0.1822	0.117	0.1111
2 kc	0.070	0.122	0.017	0.017
3 kc	0.009	0.057	$9X10^{-4}$	$7X10^{-4}$
4 kc	$4X10^{-3}$	$14X10^{-3}$	0	$1.9X10^{-5}$
5 kc	$4X10^{-4}$	$1.4X10^{-4}$	0	$1.4X10^{-6}$

Table 8-1. Experimental and Theoretical Values of Discrete Power Spectrum.

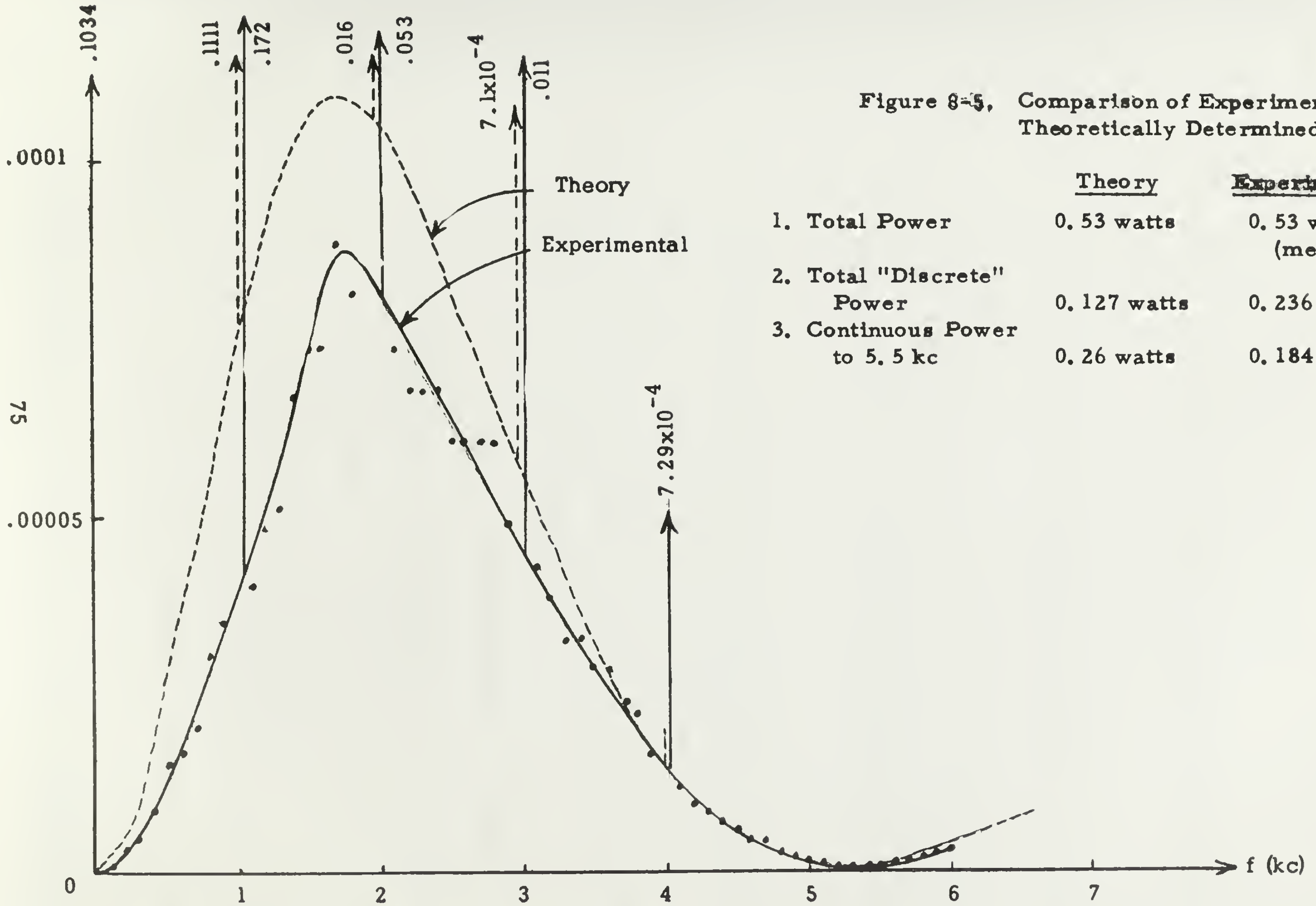


Figure 8-5. Comparison of Experimentally and Theoretically Determined Spectra.

	Theory	Experimental
1. Total Power	0.53 watts	0.53 watts (measured)
2. Total "Discrete" Power	0.127 watts	0.236 watts
3. Continuous Power to 5.5 kc	0.26 watts	0.184 watts

9. Discussion of the Results and Conclusions

A reasonable question to be answered now is "What specific statistics of the modulation process contribute to the shape of the power spectral density?" This is one question that is answered in this section. Other items discussed in this section are:

1. The "general shape" of the power spectral density of PPM waveforms.
2. The power spectral density as a function of the modulating process.
3. The parameters that determine the relative amount of the total power that is contained in the discrete and continuous spectra.
4. The effect of correlation between pulses in the pulse train with respect to their relative positions in time.
5. Some general comments on the experimental determination of power spectral density.

The shape of the power spectral densities of PPM waveforms appears to be generally of the form $(\sin x/x)^2$ if one considers only the frequencies beyond the first lobe.[1] The portion beyond the first lobe has been observed to be mostly continuous spectrum for the modulation processes considered. The reason for this can be determined from the equation upon which the computer model is based. The characteristic function

[1] It will be assumed that all pulse waveforms discussed in this section consist of rectangular pulses except when otherwise stated.

for all modulation processes considered "damps out" toward zero value as the frequency is increased, resulting in a power spectral density at higher frequencies that is essentially the same as the spectrum $S_u(f)$ of the typical sampling pulse.

The frequency at which the first lobe is terminated (where the envelope of power spectral density first becomes zero) is the reciprocal of the pulse width. The shape of the first lobe appears to be quite dependent upon the modulating process. Figures 7-5, 7-8, and 7-14 are representative of the spectrum of PPM waveforms that differ in the modulating process; i.e., a uniform process, an exponential process, and a Gaussian process, respectively. The biggest difference in these power spectral densities is in the distribution of power in the discrete frequency components.[2] A close look at the Figures 7-5, 7-8, and 7-14 additionally reveals a distinct difference in the continuous spectrum within the first lobe. As was the case with the discrete spectrum, this difference may be attributed to the type of statistical process used to modulate the PPM waveform.

The question asked at the beginning of this section will now be answered. For the cases in which statistical independence between samples is assumed, the square of the absolute value of the characteristic function is the only statistic of the modulating process that influences the power spectral density of the PPM waveform. For all cases

[2]The frequencies at which the discrete components exist are multiples of the reciprocal of the sampling rate, or the average pulse repetition rate.

considered in this investigation, the absolute value of the characteristic function is simply the Fourier transform of the probability density function of a process with zero mean. The process with a non-zero mean yields a characteristic function that is equal in magnitude to the characteristic function of the process with a zero mean, but with a phase difference. Taking the absolute value removes this phase difference, thus yielding the same results as for the process with zero mean.

For the case in which statistical independence between samples is not assumed, the power spectral density is influenced by the joint characteristic function of the modulating process. The effects of correlated samples will be discussed in greater detail in the latter part of this section.

The relative amount of the total power in the continuous spectrum of a PPM waveform with pulses of fixed duration and amplitude appears to be determined by the degree of fluctuation of the pulse shift. This observation may be confirmed by comparing figures based upon like modulating processes but with different standard deviations (or different total pulse shift permitted in the uniformly distributed case). Comparing Figures 7-11 and 7-14, for which σ_n is 0.05 and 0.1, respectively, one may observe that an additional 21.6 percent of the total power is included in the continuous spectrum for the greater fluctuating pulse shift. For the uniform case, the PPM fluctuations may be increased to the maximum level by permitting the maximum pulse excursion (delay) to be equal to the full sampling interval T_o less the pulse width u. This corresponds

to $L = T_o/(T_o - u)$ in the PPM models developed. When this is done, the discrete spectrum becomes quite small, and eventually disappears for very narrow pulse widths. The latter observation may be confirmed by direct calculations with the mathematical model; the former observation may be confirmed from Figure 7-5. In Figure 7-5, the maximum pulse excursion within a sampling interval is permitted. The pulse width is $0.1T_o$, and the aggregate power in the discrete components is reduced to less than one percent of the total power (excluding the d-c component).

Another factor which influences the relative amount of total power contained in the continuous spectrum is the pulse width. For a given modulation process, as the pulse width is decreased, the percentage of the total power contained in the continuous spectrum increases. In contrast to the considerations of the preceding paragraph, in which the total power of the PPM waveform is constant (independent of the fluctuations), if the pulse width is decreased the total power decreases unless a corresponding increase in pulse amplitude is made. A plot of the percent of total power in each component as a function of the pulse width is shown in Figure 9-1 for a single modulation process (the independent Gaussian case with $\sigma_n = 0.05$). Data for this plot was obtained from the computer model.

The effect of statistical correlation between sample values of the modulation process tends to center the continuous spectrum about the discrete components. It is interesting to note that the relative amount of the total power in the discrete components remains essentially the same

for the correlated and the uncorrelated cases considered, yet the shape of the continuous spectrum is radically changed (see Figures 7-11, 12, and 13, for example). However, in the limit as the correlation function of the modulation process approaches unity, the PPM waveform is essentially modulated by a d-c level, and the power spectral density is all at discrete frequencies. In fact, the spectrum becomes that of an unmodulated pulse train (Figure 7-4).

The correlated case is a practical consideration for typical PPM pulse trains encountered. For most PPM systems, there is time division multiplexing of several channels of data pulses, which are further interleaved with synchronizing (sync) pulses. The sync pulses are transmitted at a periodic rate. The exact periodicity of the sync pulses results in a contribution to the discrete components of the power spectral density. Further, the data pulses are normally statistically related to the sync pulses. The mean position of the data pulse is related to the sync pulse by some fixed interval for each process. The degree of correlation between data and sync pulses is set by the data pulse variance about its mean displacement. Finally, adjacent data pulses may be correlated, as was the case for the Gaussian correlated process of the simpler single channel PPM model used for this investigation.

The verification of the theoretical results by taking experimental measurements was discussed in section 8. The measurement of the continuous spectrum of the power spectral density can be quite arduous. The difficulty is partly due to the fluctuations of the modulating process

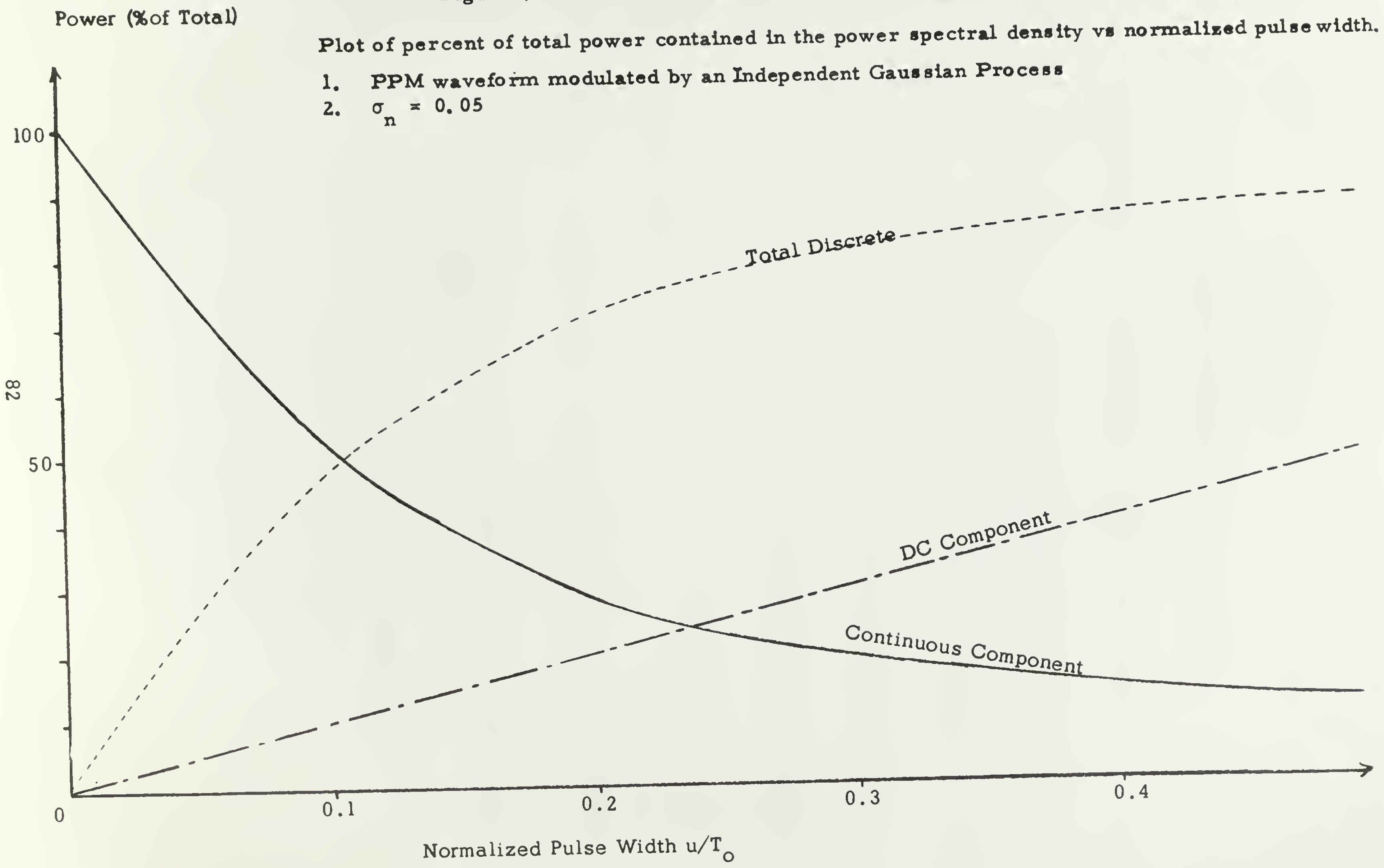
(or the degree of modulation) varying in a probabilistic manner during the measuring interval. This effect, of course, can be minimized by averaging the measurements over a longer measurement interval. Furthermore, the noise introduced by complex measuring equipment can obscure the low-level signals encountered in the continuous spectrum.

The slight discrepancy between the measured and the calculated power spectral densities displayed in Figure 8-5 may be largely attributed to the non-linearity of the pulse-position modulator for the larger modulating voltages. Consequently, the standard deviation of the pulse shift ϵ is not as large as that determined from the measured standard deviation of the modulating process. This results in a smaller continuous component and corresponding increases in the discrete components as compared to the calculated power spectral density, which was based upon a linear modulation scheme.

Figure 9-1. Breakdown of Power Components for Varying Pulse Widths.

Plot of percent of total power contained in the power spectral density vs normalized pulse width.

1. PPM waveform modulated by an Independent Gaussian Process
2. $\sigma_n = 0.05$



10. Bibliography

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APPENDIX A

The Pulse-Position Modulator

A pulse-position modulator was designed and constructed so that experimental measurements could be made on the type of PPM waveform used for the mathematical model. The design specifications were to build a device that sampled a modulating process at a periodic rate and generated a pulse train, the pulses of which being delayed in time by an amount linearly related to the sampled value. To accomplish this, it was decided to build a sampler which had as its "hold" circuit an RC network with a time constant that resulted in an approximately linear decay during the interval under consideration. This linearly decaying voltage was then compared to a "zero modulation" reference. When the two voltages were equal, a pulse was transmitted to a monostable multivibrator. The monostable multivibrator was used to give a pulse output of the desired shape, and to allow easy variation of pulse width and pulse amplitude. See Figure A-2 for pictures taken of the unmodulated and the modulated pulse trains at the multivibrator output.

The circuit diagram is shown in Figure A-1. The variable parameters of the circuit are the pulse width (change R_8), the pulse height (change V_{cc}), and the sampling rate (change R_1 and pulse rate of the pulse generator). The sampling rate is, of course, the average pulse repetition rate of the PPM waveform.

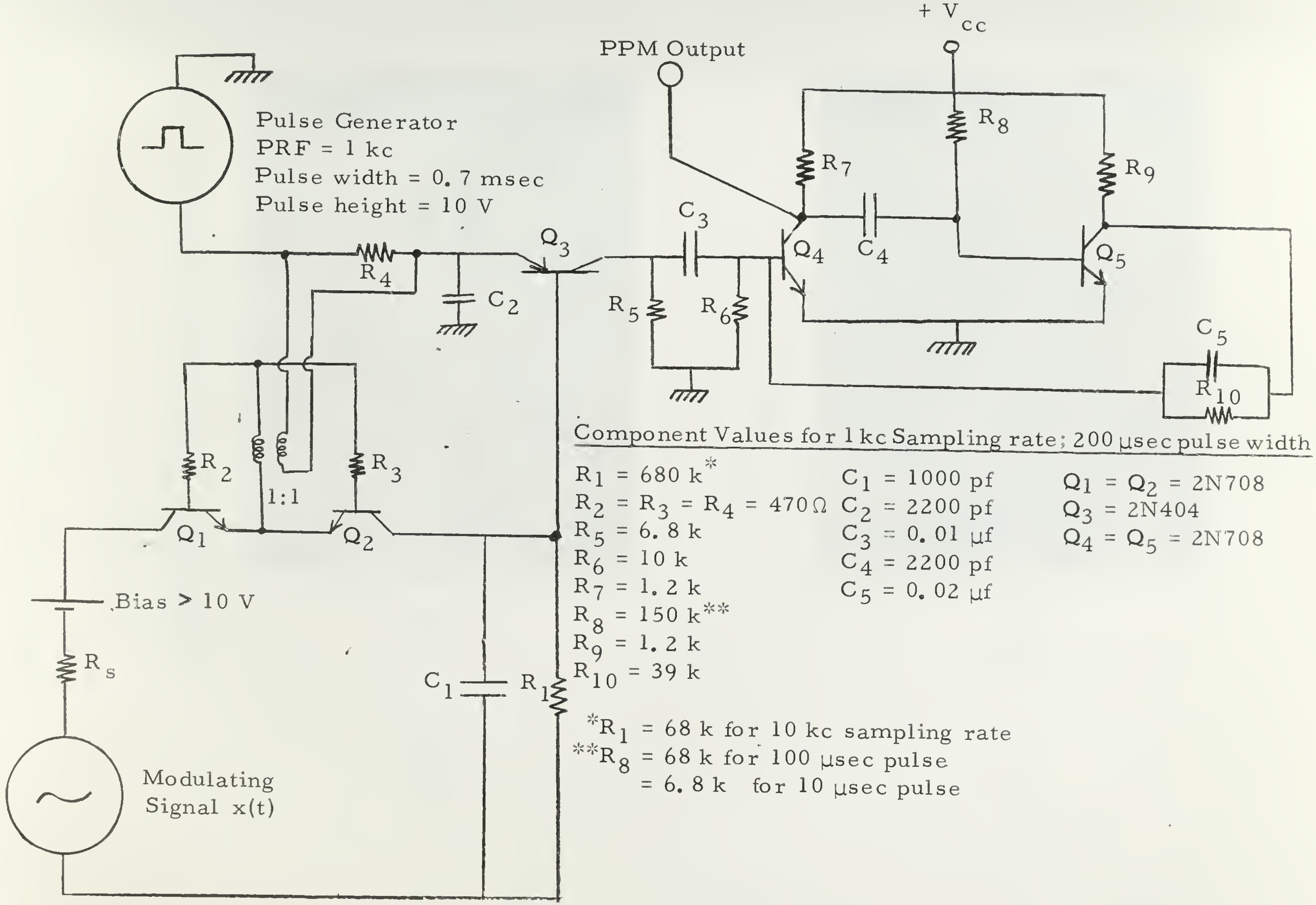
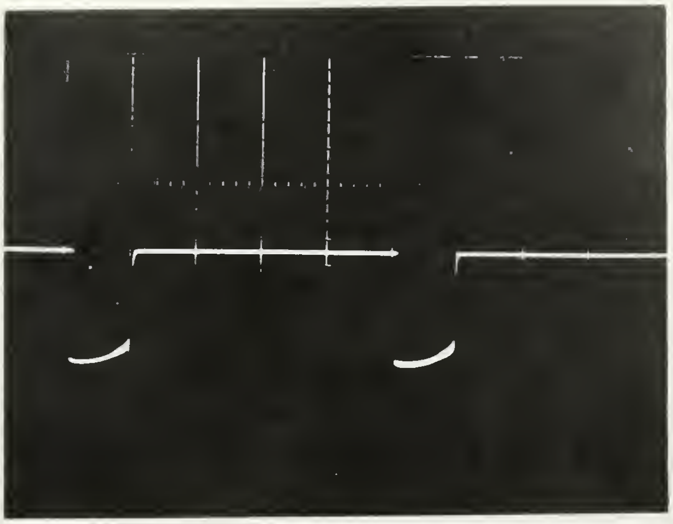
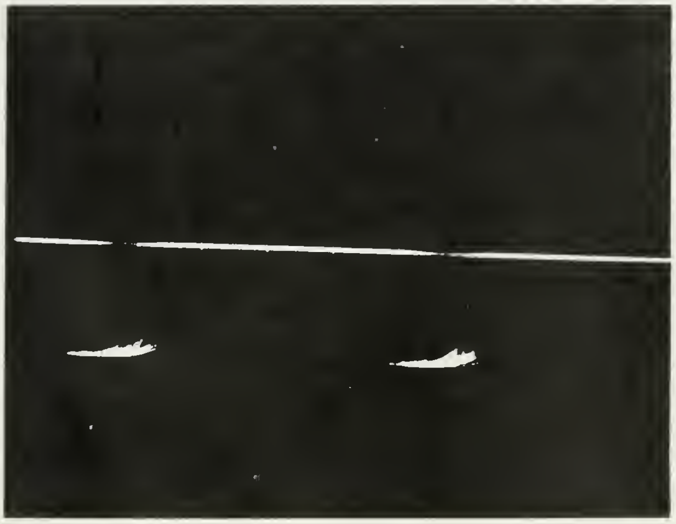


Figure A-1. Circuit Diagram of the Pulse-Position Modulator.



Horiz: 0.2 msec/cm
Vert: 1 V/cm

Unmodulated Pulse Train



Horiz: 0.2 msec/cm
Vert: 1 V/cm
$\sigma_n = 0.08$

Pulse train modulated with wideband Gaussian noise.

Figure A-2. Performance of the Pulse-Position Modulator.